LIFTING NON-PROPER TROPICAL INTERSECTIONS

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ABSTRACT. We prove that if X, X' are closed subschemes of a torus $\mathbf{T}$ over a non-Archimedean field K, of complementary codimension and with finite intersection, then the stable tropical intersection along a (possibly positive-dimensional, possibly unbounded) connected component C of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ lifts to algebraic intersection points, with multiplicities. This theorem requires potentially passing to a suitable toric variety $X(\Delta)$ and its associated extended tropicalization $N_{\mathbf{R}}(\Delta)$; the algebraic intersection points lifting the stable tropical intersection will have tropicalization somewhere in the closure of C in $N_{\mathbf{R}}(\Delta)$. The proof involves a result on continuity of intersection numbers in the context of non-Archimedean analytic spaces.

1. INTRODUCTION

Let K be a field equipped with a nontrivial[1] non-Archimedean valuation $\mathrm{val} : K \to \mathbf{R} \cup \{\infty\}$, and suppose that K is complete or algebraically closed. Let $\mathbf{T} \cong \mathbf{G}_m^n$ be a finite-rank split torus over K with coordinate functions $x_1, \ldots, x_n$. The *tropicalization* map is the function $\mathrm{trop} : |\mathbf{T}| \to \mathbf{R}^n$ given by $\mathrm{trop}(\xi) = (\mathrm{val}(x_1(\xi)), \ldots, \mathrm{val}(x_n(\xi)))$, where $|X|$ denotes the set of closed points of a scheme X. Given a closed subscheme $X \subseteq \mathbf{T}$, the *tropicalization* of X is the closure (with respect to the Euclidean topology) of the set $\mathrm{trop}(|X|)$ in $\mathbf{R}^n$, and is denoted $\mathrm{Trop}(X)$. This is a subset which can be endowed with the structure of a weighted polyhedral complex of the same dimension as X. In particular, it is a combinatorial object, a "shadow" of X which is often much easier to analyze than X itself. It is therefore important that one can recover information about X from its tropicalization.

An example of this idea is to relate the intersection of X with a second closed subscheme $X' \subseteq \mathbf{T}$ to the intersections of their tropicalizations. One might hope that $\mathrm{Trop}(X \cap X') = \mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, but this is not generally the case. For example, let K be the field of Puiseux series over $\mathbf{C}$ with uniformizer t. The curves $X = \{x + y = 1\}$ and $X' = \{tx + y = 1\}$ do not meet in $\mathbf{G}_m^2$, but $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ is the ray $\mathbf{R}_{\geq 0} \cdot (1, 0) \subset \mathbf{R}^2$. This example is "degenerate" in the sense that $\mathrm{Trop}(X)$ does not intersect $\mathrm{Trop}(X')$ transversely; generically the intersection of two one-dimensional polyhedral complexes in $\mathbf{R}^2$ is a finite set of points. This is in fact the only obstruction: assuming X, X' pure dimensional, if $\mathrm{Trop}(X)$ meets $\mathrm{Trop}(X')$ in the expected codimension at a point $v \in \mathbf{R}^n$, then $v \in \mathrm{Trop}(X \cap X')$. This was proved by Osserman and Payne, who in fact prove much more: they show that in a suitable sense, the tropicalization of the intersection cycle $X \cdot X'$ is equal to the stable tropical intersection $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$, still under the hypothesis that $\mathrm{Trop}(X)$ meets $\mathrm{Trop}(X')$ in the expected codimension; see [OP10, Theorem 1.1, Corollary 5.1.2]. In particular, if $\mathrm{codim}(X) + \mathrm{codim}(X') = \dim(\mathbf{T})$ and $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ is a finite set of points, then $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$ is a weighted sum of points of $\mathbf{R}^n$; these points then lift, with multiplicities, to points of $X \cdot X'$. Hence in this case one can compute local intersection numbers via tropicalization.

This paper will be concerned with the case when $\mathrm{codim}(X) + \mathrm{codim}(X') = \dim(\mathbf{T})$, but when the intersection $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ may have higher-dimensional connected components. The stable tropical intersection $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$ is still a well-defined finite set of points contained in $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, obtained by translating $\mathrm{Trop}(X)$ by a generic vector v and then taking the limit as $v \to 0$, but it is no longer the case that $\mathrm{Trop}(X \cdot X') = \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$. Indeed, in the above example of $X = \{x + y = 1\}$ and $X' = \{tx + y = 1\}$, the stable tropical intersection is the point $(0, 0)$ with multiplicity 1, but $X \cap X' = \emptyset$. This illustrates the need to compactify the situation in the direction of

[1]In the introduction we assume that the valuation is nontrivial for simplicity; we will prove the main theorems in the trivially valued case as well.

the ray $\mathbf{R}_{\geq 0} \cdot (1,0)$. Let us view X, X' as curves in $\mathbf{A}^1 \times \mathbf{G}_m$, and extend the tropicalization map to a map $\operatorname{trop} : |\mathbf{A}^1 \times \mathbf{G}_m| \to (\mathbf{R} \cup \{\infty\}) \times \mathbf{R}$ in the obvious way. Then $X \cap X'$ is the reduced point $(0,1)$, and $\operatorname{Trop}(X \cap X') = \{(\infty, 0)\}$ is contained in the closure of $\operatorname{Trop}(X) \cap \operatorname{Trop}(X')$ in $(\mathbf{R} \cup \{\infty\}) \times \mathbf{R}$. It is not a coincidence that the multiplicity of the point $(0,1) \in X \cap X'$ coincides with the multiplicity of $(0,0) \in \operatorname{Trop}(X) \cdot \operatorname{Trop}(X')$: we have lost the ability to pinpoint the exact location of the point $\operatorname{Trop}(X \cap X')$ beyond saying that it lies in the closure of $\operatorname{Trop}(X) \cap \operatorname{Trop}(X')$, but we are still able to recover its multiplicity using the stable tropical intersection.

In order to carry out this strategy in general, we need to make precise the notion of "compactifying in the directions where the tropicalization is infinite": we say that an integral pointed fan Δ is a *compactifying fan* for a polyhedral complex Π provided that the recession cone of each cell of Π is a union of cones in Δ. The setup for the main theorem is then as follows. Let $X_1, \ldots, X_m \subseteq \mathbf{T}$ be pure-dimensional closed subschemes with $\sum_{i=1}^{m} \operatorname{codim}(X_i) = \dim(\mathbf{T})$, and let $C \subseteq \bigcap_{i=1}^{m} \operatorname{Trop}(X_i)$ be a connected component. Let Π be the polyhedral complex underlying C (with respect to some choice of polyhedral complex structures on the $\operatorname{Trop}(X_i)$), and let Δ be a compactifying fan for Π. Let M be the lattice of characters of $\mathbf{T}$ and let N be its dual lattice, so the $\operatorname{Trop}(X_i)$ naturally live in $N_\mathbf{R} = N \otimes_\mathbf{Z} \mathbf{R}$. We partially compactify the torus with the toric variety $X(\Delta)$, which contains $\mathbf{T}$ as a dense open subscheme. The extended tropicalization is a topological space $N_\mathbf{R}(\Delta)$ which canonically contains $N_\mathbf{R}$ as a dense open subset, and which is equipped with a map $\operatorname{trop} : |X(\Delta)| \to N_\mathbf{R}(\Delta)$ extending $\operatorname{trop} : |\mathbf{T}| \to N_\mathbf{R}$; see (2.4). Let $\overline{C}$ be the closure of C in $N_\mathbf{R}(\Delta)$; this is a compact set since Δ is a compactifying fan for Π (Remark 3.3). For an isolated point $\xi \in \bigcap_{i=1}^{m} \overline{X}_i$ we let $i_K(\xi, \overline{X}_1 \cdots \overline{X}_m; X(\Delta))$ denote the multiplicity of ξ in the intersection class $\overline{X}_1 \cdots \overline{X}_m$, and for $v \in \bigcap_{i=1}^{m} \operatorname{Trop}(X_i)$ we let $i(v, \prod_{i=1}^{m} \operatorname{Trop}(X_i))$ denote the multiplicity of v in the stable tropical intersection $\operatorname{Trop}(X_1) \cdots \operatorname{Trop}(X_m)$.

Theorem. *If $X(\Delta)$ is smooth, and if there are only finitely many points of $|\overline{X}_1 \cap \cdots \cap \overline{X}_m|$ mapping to $\overline{C}$ under* trop*, then*

$$\sum_{\substack{\xi \in |\bigcap_{i=1}^{m} \overline{X}_i| \\ \operatorname{trop}(\xi) \in \overline{C}}} i_K\big(\xi, \overline{X}_1 \cdots \overline{X}_m; X(\Delta)\big) = \sum_{v \in C} i\big(v, \operatorname{Trop}(X_1) \cdots \operatorname{Trop}(X_m)\big).$$

See Theorem 6.10. This can be seen as a lifting theorem for points in the stable tropical intersection $\operatorname{Trop}(X_1) \cdots \operatorname{Trop}(X_m)$, with the provisos that we may have to do some compactification of the situation first, and that the tropicalizations of the points of the algebraic intersection $\overline{X}_1 \cdots \overline{X}_m$ corresponding to a point v of the stable tropical intersection are only confined to the closure of the connected component of $\bigcap_{i=1}^{m} \operatorname{Trop}(X_i)$ containing v.

The finiteness assumption on $\bigcap_{i=1}^{m} \overline{X}_i$ is also necessary in this generality — we will provide some conditions under which it is automatically satisfied. In particular, when C is *bounded*, the compactifying fan is unnecessary, and we have:

Corollary. *Suppose that C is bounded. Then there are only finitely many points of $|X_1 \cap \cdots \cap X_m|$ mapping to C under* trop*, and*

$$\sum_{\substack{\xi \in |\bigcap_{i=1}^{m} X_i| \\ \operatorname{trop}(\xi) \in C}} i_K\big(\xi, X_1 \cdots X_m; \mathbf{T}\big) = \sum_{v \in C} i\big(v, \operatorname{Trop}(X_1) \cdots \operatorname{Trop}(X_m)\big).$$

The proof of the main Theorem proceeds as follows. Assume for simplicity that K is both complete and algebraically closed. Let $X, X' \subseteq \mathbf{T}$ be pure-dimensional closed subschemes with $\operatorname{codim}(X) + \operatorname{codim}(X') = \dim(\mathbf{T})$, and let C be a connected component of $\operatorname{Trop}(X) \cap \operatorname{Trop}(X')$. Assume for the moment that C is bounded. Let $v \in N$ be a generic cocharacter, regarded as a homomorphism $v : \mathbf{G}_m \to \mathbf{T}$. Then $(\operatorname{Trop}(X) + \varepsilon \cdot v) \cap \operatorname{Trop}(X')$ is a finite set for small enough ε, and the stable tropical intersection is equal to $\lim_{\varepsilon \to 0} (\operatorname{Trop}(X) + \varepsilon \cdot v) \cap \operatorname{Trop}(X')$; this can be seen as a "continuity of local tropical intersection numbers". For $t \in K^\times$ the tropicalization of $v(t) \cdot X$ is equal to $\operatorname{Trop}(X) - \operatorname{val}(t) \cdot$

v, so for small nonzero values of $\mathrm{val}(t)$ we can apply Osserman-Payne's tropical lifting theorem to $(v(t) \cdot X) \cap X'$. Hence what we want to prove is a theorem on continuity of *local* algebraic intersection numbers that applies to the family $\mathfrak{Y} = \{(v(t) \cdot X) \cdot X'\}_{\mathrm{val}(t) \in [-\varepsilon, \varepsilon]}$.

There are two problems with proving this continuity of local intersection numbers, both of which have the same solution. The first is that the base of the family $\mathfrak{Y}$ is the set $\mathcal{S}_\varepsilon(K) = \{t \in K^\times : \mathrm{val}(t) \in [-\varepsilon, \varepsilon]\}$, which is not algebraic but an *analytic* annulus in $\mathbf{G}_m$. The second is that we only want to count intersection multiplicities in a neighborhood of C — more precisely, if P is a polytope containing C in its interior and disjoint from the other components of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, then for every $t \in \mathcal{S}_\varepsilon(K)$ we only want to count intersection multiplicities of points in $\mathcal{U}_P(K) = \mathrm{trop}^{-1}(P)$, which is again an analytic subset of $\mathbf{T}$. Therefore we will prove that dimension-zero intersection numbers of analytic spaces are constant in flat families over an analytic base. This is one of the main ideas of the paper; the other idea, orthogonal to this one, is the precise compactification procedure described above, which is necessary when C is unbounded.

Outline of the paper. Many of the technical difficulties in this paper revolve around the need to pass to a compactifying toric variety when our connected component C of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ is unbounded. As such, section 3 is devoted to introducing compactifying and compatible fans Δ, and studying the behavior of the closure operation for polyhedra in $N_{\mathbf{R}}(\Delta)$. The main result is Proposition 3.12, which says in particular that for a suitable fan Δ, the extended tropicalization of the intersection of the closures of X and X' in $X(\Delta)$ is contained in the closure of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, and that the same can be achieved for individual connected components of the intersection. This is quite important in the statement of the Theorem above, since we want to sum over all closed points ξ of $\overline{X} \cap \overline{X}'$ with $\mathrm{trop}(\xi) \in \overline{C}$, and is also vital in section 4.

In section 4 we prove a version of the tropical moving lemma: the stable tropical intersection $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X')$ is defined locally by translating $\mathrm{Trop}(X)$ by a small amount in the direction of a generic displacement vector v, and in Lemma 4.7 we make these conditions precise. The main point of section 4, however, is to show that for $v \in N$ satisfying the tropical moving lemma, the corresponding family $\{(v(t) \cdot \overline{X}) \cap \overline{X}' \cap \mathcal{U}_P\}_{t \in \mathcal{S}_\varepsilon}$ of analytic subspaces of $\mathcal{U}_P$, where P is a polyhedral neighborhood of C, is *proper* over $\mathcal{S}_\varepsilon$. See Proposition 4.19. We therefore give a brief discussion of the analytic notion of properness in section 4, which we conclude with the very useful tropical criterion for properness of a family of analytic subspaces of a toric variety (Proposition 4.16).

In section 5 we define local intersection multiplicities of dimension-zero intersections of analytic spaces in a smooth ambient space, using a slight modification of Serre's definition. These analytic intersection numbers coincide with the algebraic ones in the case of analytifications of closed subschemes (Proposition 5.7). The main result (Proposition 5.8) is the continuity of analytic intersection numbers mentioned above: if $\mathcal{X}, \mathcal{X}'$ are analytic spaces, flat over a connected base $\mathcal{S}$, inside a smooth analytic space $\mathcal{Z}$, such that $\mathcal{X} \cap \mathcal{X}'$ is finite over $\mathcal{S}$, then the total intersection multiplicities on any two fibers are equal.

In section 6 we prove the main theorem (Theorem 6.4) and its corollaries, combining the results of sections 4 and 5. We also treat the case of intersecting more than two subschemes of $\mathbf{T}$ by reducing to intersection with the diagonal. We conclude by giving a detailed worked example in section 7.

2. Analytifications and tropicalizations

We will use the following general notation throughout the paper. If P is a subspace of a topological space X, its interior (resp. closure) in X will be denoted P° (resp. $\overline{P}$). If $f : X \to Y$ is a map (of sets, schemes, analytic spaces, etc.) the fiber over $y \in Y$ will be denoted $X_y = f^{-1}(y)$.

By a *cone* in a Euclidean space we will always mean a polyhedral cone.

2.1. Non-Archimedean fields. We fix a non-Archimedean field K, i.e. a field equipped with a non-Archimedean valuation $\mathrm{val} : K \to \mathbf{R} \cup \{\infty\}$. We will assume throughout that K is complete or algebraically closed, and except in (2.3), (2.4), and section 6, we assume further that val is nontrivial and that K is complete with respect to val, in order to be able to work with analytic spaces over K. Let

$|\cdot| = \exp(-\operatorname{val}(\cdot))$ be the corresponding absolute value and let $G = \operatorname{val}(\overline{K}^\times) \subseteq \mathbf{R}$ be the saturation of the value group of K.

By a *valued field extension* of K we mean a non-Archimedean field K' equipped with an embedding $K \hookrightarrow K'$ which respects the valuations.

2.2. Analytic spaces. Assume that K is complete and nontrivially valued.[2] In this paper, by an *analytic space* we mean a separated (i.e. the underlying topological space is Hausdorff), good, strictly K-analytic space in the sense of [Ber93]. In particular, all K-affinoid algebras and K-affinoid spaces are assumed to be strictly K-affinoid. We will generally use calligraphic letters to refer to analytic spaces. For a K-affinoid algebra A, its Berkovich spectrum $\mathscr{M}(A)$ is an analytic space whose underlying topological space is the set of bounded multiplicative semi-norms $\|\cdot\| : A \to \mathbf{R}_{\geq 0}$, equipped with topology of pointwise convergence. An affinoid space is compact. If $\mathcal{X}$ is an analytic space, $|\mathcal{X}|$ will denote the set of classical "rigid" points of $\mathcal{X}$; this definition is local on $\mathcal{X}$, and if $\mathcal{X} = \mathscr{M}(A)$ is affinoid, then $|\mathcal{X}|$ is naturally identified with the set of maximal ideals of A. The subset $|\mathcal{X}|$ is everywhere dense in $\mathcal{X}$ by [Ber90, Proposition 2.1.15]. We also let $\mathcal{X}(\overline{K}) = \varinjlim_{K'} \mathcal{X}(K')$, where $\mathcal{X}(K') = \operatorname{Hom}_K(\mathscr{M}(K'), \mathcal{X})$ and the union runs over all finite extensions K' of K. There is a natural surjective map $\mathcal{X}(\overline{K}) \twoheadrightarrow |\mathcal{X}|$.

For a point x of an analytic space $\mathcal{X}$, we let $\mathscr{H}(x)$ denote the completed residue field at x. This is a complete valued field extension of K which plays the role of the residue field at a point of a scheme. In particular, if $\mathcal{Y} \to \mathcal{X}$ is a morphism, then the set-theoretic fiber $\mathcal{Y}_x$ is naturally an $\mathscr{H}(x)$-analytic space. The point x is rigid if and only if $[\mathscr{H}(x) : K] < \infty$, in which case we will generally use the notation $K(x) = \mathscr{H}(x)$.

Let $\mathcal{X}$ be an analytic space. An *analytic domain* in $\mathcal{X}$ is, roughly, a subset $\mathcal{Y}$ which naturally inherits the structure of analytic space from $\mathcal{X}$. These play the role of the open subschemes of a scheme; in particular, any open subset of $\mathcal{X}$ is an analytic domain. An analytic domain need not be open, however; for example, an *affinoid domain* in $\mathcal{X}$ is an analytic domain which is also an affinoid space (which is compact, hence closed). A *Zariski-closed subspace* of $\mathcal{X}$ is an analytic space $\mathcal{Y} \hookrightarrow \mathcal{X}$ which is locally defined by the vanishing of some number of analytic functions on $\mathcal{X}$. The set underlying $\mathcal{Y}$ is closed in $\mathcal{X}$.

For any separated, finite-type K-scheme X we let X^{an} denote the analytification of X. This analytic space comes equipped with a map of ringed spaces $X^{\mathrm{an}} \to X$ which identifies the set $|X|$ of closed points (resp. the set $X(\overline{K})$ of geometric points) with $|X^{\mathrm{an}}|$ (resp. $X^{\mathrm{an}}(\overline{K})$). If for $x \in |X|$ we let $K(x)$ denote the residue field at x, then $K(x)$ is identified with the completed residue field of the associated point $x \in |X^{\mathrm{an}}|$. The analytification functor respects all fiber products and complete valued extensions of the ground field. In the case that $X = \operatorname{Spec}(A)$ is affine, we will identify the topological space underlying X^{an} with the space of all multiplicative semi-norms $\|\cdot\| : A \to \mathbf{R} \cup \{\infty\}$ extending the absolute value on K.

If X is a K-scheme (resp. a K-analytic space) and K' is a field extension (resp. complete valued field extension) of K, we let $X_{K'}$ denote the base change to K'.

2.3. Tropicalization. Here we assume that K is a complete or algebraically closed, possibly trivially-valued non-Archimedean field. Let $M \cong \mathbf{Z}^n$ be a finitely generated free abelian group and $N = \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$ its dual. For any subgroup $\Gamma \subseteq \mathbf{R}$ we let $M_\Gamma = M \otimes_{\mathbf{Z}} \Gamma$ and $N_\Gamma = N \otimes_\Gamma \mathbf{Z} = \operatorname{Hom}_{\mathbf{Z}}(M, \Gamma)$. Let $\mathbf{T} = \operatorname{Spec}(K[M])$ be the torus with character lattice M. Given a closed subscheme $X \subseteq \mathbf{T}$ and a point $v \in N_{\mathbf{R}}$, the *initial degeneration* $\operatorname{in}_v(X)$ is a canonically defined scheme over the residue field of K, of finite type if $v \in N_G \subseteq N_{\mathbf{R}}$. The *tropicalization* of X is the subset $\operatorname{Trop}(X) \subseteq N_{\mathbf{R}}$ of all v such that $\operatorname{in}_v(X)$ is nonempty. If K'/K is a complete or algebraically closed valued field extension then $\operatorname{Trop}(X_{K'}) = \operatorname{Trop}(X)$. The set $\operatorname{Trop}(X)$ can be enriched with the structure of a polyhedral complex (which is in general non-canonical) with the property that if $v, v' \in N_G$ lie in the interior of the same cell, then $\operatorname{in}_v(X_{\overline{K}}) \cong \operatorname{in}_{v'}(X_{\overline{K}})$. This polyhedral complex has positive integer weights canonically

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[2]For convenience, in this paper we only work with analytic spaces over nontrivially valued fields, although Berkovich's theory is valid in the trivially valued case.

assigned to each facet, defined as follows: let $P \subset \mathrm{Trop}(X)$ be a facet, let $v \in \mathrm{relint}(P)$, and let K' be an algebraically closed valued field extension of K with value group G' such that $v \in N_{G'}$. The *tropical multiplicity* $m(P)$ of P is defined to be the sum of the multiplicities of the irreducible components of $\mathrm{in}_v(X_{K'})$. This is independent of the choice of K' by [OP10, Remark A.5], or [BPR11, §4.18] in the complete case (see also [BPR11, Lemma 4.19]). The weights are insensitive to algebraically closed valued field extensions. See for instance [OP10, §2] for a more detailed survey of the above.

If $X = V(f)$ is the hypersurface defined by a Laurent polynomial $f \in K[M]$ then we write $\mathrm{Trop}(f) = \mathrm{Trop}(X)$; the set $\mathrm{Trop}(f)$ is equipped with a canonical weighted polyhedral complex structure. See for instance [Rab10, §8].

For $u \in M$, let $x^u \in K[M]$ denote the corresponding character. The *tropicalization map* $\mathrm{trop} : |\mathbf{T}| \to N_{\mathbf{R}}$ is the map defined by $\langle u, \mathrm{trop}(\xi) \rangle = -\mathrm{val}(x^u(\xi))$, where $\langle \cdot, \cdot \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \to \mathbf{R}$ is the canonical pairing. We also denote the composition $\mathbf{T}(\overline{K}) \to |\mathbf{T}| \to N_{\mathbf{R}}$ by trop. Note that this definition only makes sense when K is complete or algebraically closed, as $\mathrm{val}(x^u(\xi))$ is not in general well-defined if K is neither. If K is complete and nontrivially valued, we define $\mathrm{trop} : \mathbf{T}^{\mathrm{an}} \to N_{\mathbf{R}}$ by $\langle u, \mathrm{trop}(\|\cdot\|) \rangle = \log(\|x^u\|)$; this is a continuous, proper surjection which is compatible with $\mathrm{trop} : |\mathbf{T}| \to N_{\mathbf{R}}$ under the identification $|\mathbf{T}| = |\mathbf{T}^{\mathrm{an}}|$.

Let $X \subseteq \mathbf{T}$ be a closed subscheme. If K is nontrivially valued then $\mathrm{Trop}(X)$ is the closure of $\mathrm{trop}(|X|)$ in $N_{\mathbf{R}}$, and if in addition K is complete then $\mathrm{Trop}(X) = \mathrm{trop}(X^{\mathrm{an}})$. If K is trivially valued then $\mathrm{trop}(|X|) = \{0\}$ or is empty.

2.4. Extended tropicalization.

We continue to assume that K is a complete or algebraically closed, possibly trivially-valued non-Archimedean field. If σ is an integral cone in $N_{\mathbf{R}}$, we let $X(\sigma)$ denote the affine toric variety $\mathrm{Spec}(K[\sigma^\vee \cap M])$, where

$$\sigma^\vee = \{ u \in M_{\mathbf{R}} \ : \ \langle u, v \rangle \leq 0 \text{ for all } v \in \sigma \}.$$

Likewise for an integral fan Δ in $N_{\mathbf{R}}$ we let $X(\Delta)$ be the toric variety obtained by gluing the affine toric varieties $X(\sigma)$ for $\sigma \in \Delta$.

Let σ be an integral cone in $N_{\mathbf{R}}$. We define $N_{\mathbf{R}}(\sigma) = \mathrm{Hom}_{\mathbf{R}_{\geq 0}}(\sigma^\vee, \mathbf{R} \cup \{-\infty\})$, the set of homomorphisms of additive monoids with an action of $\mathbf{R}_{\geq 0}$. We equip $N_{\mathbf{R}}(\sigma)$ with the topology of pointwise convergence. The tropicalization map extends to a map $\mathrm{trop} : |X(\sigma)| \to N_{\mathbf{R}}(\sigma)$, again using the formula $\langle u, \mathrm{trop}(\xi) \rangle = -\mathrm{val}(x^u(\xi))$. If K is complete and nontrivially valued then we define $\mathrm{trop} : X(\sigma)^{\mathrm{an}} \to N_{\mathbf{R}}(\sigma)$ by $\langle u, \mathrm{trop}(\|\cdot\|) \rangle = \log(\|x^u\|)$; as above this is a continuous, proper surjection which is compatible with $\mathrm{trop} : |X(\sigma)| \to N_{\mathbf{R}}(\sigma)$ under the identification $|X(\sigma)| = |X(\sigma)^{\mathrm{an}}|$. If Δ is an integral fan in $N_{\mathbf{R}}$ we set $N_{\mathbf{R}}(\Delta) = \bigcup_{\sigma \in \Delta} N_{\mathbf{R}}(\sigma)$; the tropicalization maps $\mathrm{trop} : |X(\sigma)| \to N_{\mathbf{R}}(\sigma)$ (resp. $\mathrm{trop} : X(\sigma)^{\mathrm{an}} \to N_{\mathbf{R}}(\sigma)$ in the complete nontrivially valued case) glue to give a map $\mathrm{trop} : |X(\Delta)| \to N_{\mathbf{R}}(\Delta)$ (resp. a continuous, proper surjection $\mathrm{trop} : X(\Delta)^{\mathrm{an}} \to N_{\mathbf{R}}(\Delta)$). As above we also use trop to denote the composite map $X(\Delta)(\overline{K}) \to |X(\Delta)| \to N_{\mathbf{R}}(\Delta)$.

There is a natural decomposition $N_{\mathbf{R}}(\Delta) = \coprod_{\sigma \in \Delta} N_{\mathbf{R}}/\mathrm{span}(\sigma)$, which respects the decomposition of $X(\Delta)$ into torus orbits. We will make this identification implicitly throughout the paper. If $X(\Delta) = X(\sigma)$ is an affine toric variety, a monoid homomorphism $v : \sigma^\vee \to \mathbf{R} \cup \{-\infty\}$ is in the stratum $N_{\mathbf{R}}/\mathrm{span}(\tau)$ if and only if $v^{-1}(\mathbf{R}) = \tau^\perp \cap \sigma^\vee$. For a cone $\sigma \subseteq N_{\mathbf{R}}$ we let π_σ denote the quotient map $N_{\mathbf{R}} \to N_{\mathbf{R}}/\mathrm{span}(\sigma)$. We will use the following explicit description of the topology on $N_{\mathbf{R}}(\sigma)$:

Lemma 2.5. *Let $\sigma \subset N_{\mathbf{R}}$ be a pointed cone. A sequence $v_1, v_2, \ldots \in N_{\mathbf{R}}$ converges to the point $\overline{v} \in N_{\mathbf{R}}/\mathrm{span}(\tau) \subseteq N_{\mathbf{R}}(\sigma)$ for some $\tau \prec \sigma$ if and only if both of the following hold:*

(1) *$\langle u, v_i \rangle \to \langle u, \overline{v} \rangle$ as $i \to \infty$ for all $u \in \sigma^\vee \cap \tau^\perp$ (equivalently, $\pi_\tau(v_i) \to \overline{v}$ as $i \to \infty$), and*

(2) *$\langle u, v_i \rangle \to -\infty$ as $i \to \infty$ for all $u \in \sigma^\vee \setminus \tau^\perp$.*

Proof. Since $N_{\mathbf{R}}(\sigma)$ is equipped with the topology of pointwise convergence, this follows immediately from the fact that for $u \in \sigma^\vee$ we have $\langle u, \overline{v} \rangle \neq -\infty$ if and only if $u \in \tau^\perp$ (note that since σ is pointed, $\sigma^\vee$ spans $M_{\mathbf{R}}$). ∎

If $X \subseteq X(\Delta)$ is a closed subscheme, its *extended tropicalization* $\mathrm{Trop}(X, \Delta) \subset N_{\mathbf{R}}(\Delta)$ can be defined by tropicalizing each torus orbit separately. If the valuation on K is nontrivial then $\mathrm{Trop}(X, \Delta)$

is the closure of $\mathrm{trop}(|X|)$ in $N_{\mathbf{R}}(\Delta)$, and if in addition K is complete then $\mathrm{Trop}(X, \Delta) = \mathrm{trop}(X^{\mathrm{an}})$. See [Pay09, Rab10] for details on extended tropicalizations.

3. COMPATIBLE AND COMPACTIFYING FANS

If $\mathcal{P}$ is any finite collection of polyhedra, its *support* is the closed subset $|\mathcal{P}| = \bigcup_{p \in \mathcal{P}} P$. In this section we develop the related notions of compatible and compactifying fans for $\mathcal{P}$. Roughly, if Δ is compatible with $\mathcal{P}$ then the closure of $|\mathcal{P}|$ is easy to calculate in $N_{\mathbf{R}}(\Delta)$, and if Δ is a compactifying fan then the closure of $|\mathcal{P}|$ in $N_{\mathbf{R}}(\Delta)$ is compact — i.e., $N_{\mathbf{R}}(\Delta)$ compactifies $N_{\mathbf{R}}$ in the directions in which $|\mathcal{P}|$ is infinite. This will be important when $|\mathcal{P}|$ is a connected component of the intersection of tropicalizations.

The *recession cone* of a polyhedron $P \subseteq N_{\mathbf{R}}$ is defined to be the set

$$\rho(P) = \{w \in N_{\mathbf{R}} \ : \ v + w \in P \text{ for all } v \in P\}.$$

If P is cut out by conditions $\langle u_i, v \rangle \leq c_i$ for $u_1, \dots, u_m \in M_{\mathbf{R}}$ and $c_1, \dots, c_m \in \mathbf{R}$, then $\rho(P)$ is given explicitly by $\langle u_i, v \rangle \leq 0$ for $i = 1, \dots, m$.

Definition 3.1. Let $\mathcal{P}$ be a finite collection of polyhedra in $N_{\mathbf{R}}$ and let Δ be a pointed fan.

(1) The fan Δ is said to be *compatible with* $\mathcal{P}$ provided that, for all $P \in \mathcal{P}$ and all cones $\sigma \in \Delta$, either $\sigma \subseteq \rho(P)$ or $\mathrm{relint}(\sigma) \cap \rho(P) = \emptyset$.

(2) The fan Δ is said to be a *compactifying fan for* $\mathcal{P}$ provided that, for all $P \in \mathcal{P}$, the recession cone $\rho(P)$ is a union of cones in Δ.

The reason that we will generally require our fans to be pointed is due to the fact that if Δ is a pointed fan in $N_{\mathbf{R}}$, then $N_{\mathbf{R}}$ is canonically identified with the open subspace $N_{\mathbf{R}}(\{0\})$ of $N_{\mathbf{R}}(\Delta)$.

Following are some basic properties of compatible and compactifying fans, which are easily checked directly from the definitions.

Proposition 3.2. *Let $\mathcal{P}$ be a finite collection of polyhedra in $N_{\mathbf{R}}$.*

(1) *A compactifying fan for $\mathcal{P}$ is compatible with $\mathcal{P}$.*

(2) *A subfan of a fan compatible with $\mathcal{P}$ is compatible with $\mathcal{P}$.*

(3) *A refinement of a fan compatible with $\mathcal{P}$ is compatible with $\mathcal{P}$, and a refinement of a compactifying fan for $\mathcal{P}$ is a compactifying fan for $\mathcal{P}$.*

(4) *If a fan is compatible with $\mathcal{P}$, it is compatible with any subset of $\mathcal{P}$. A compactifying fan for $\mathcal{P}$ is a compactifying fan for any subset of $\mathcal{P}$.*

(5) *Suppose that $\mathcal{P}$ is a subset of the cells of a polyhedral complex Π, and $\mathcal{P}$ contains all the maximal cells of Π (equivalently, $\mathcal{P}$ and Π have the same support). Then a fan is compatible with $\mathcal{P}$ if and only if it is compatible with Π, and a fan is a compactifying fan for $\mathcal{P}$ if and only if it is a compactifying fan for Π.*

(6) *Let $\mathcal{P}'$ be a second finite collection of polyhedra in $N_{\mathbf{R}}$. A fan compatible with both $\mathcal{P}$ and $\mathcal{P}'$ is compatible with $\mathcal{P} \cap \mathcal{P}'$, and a compactifying fan for both $\mathcal{P}$ and $\mathcal{P}'$ is a compactifying fan for $\mathcal{P} \cap \mathcal{P}'$.*

Here the notation $\mathcal{P} \cap \mathcal{P}'$ means the set of intersections of pairs of polyhedra in $\mathcal{P}$ and $\mathcal{P}'$. If $\mathcal{P}$ and $\mathcal{P}'$ are the sets of cells of polyhedral complexes Π and Π', then $\mathcal{P} \cap \mathcal{P}'$ is not generally equal to $\Pi \cap \Pi'$, as it does not have to contain every face of every polyhedron. However, according to Proposition 3.2(5) above, a fan is compatible with $\mathcal{P} \cap \mathcal{P}'$ if and only it is compatible with $\Pi \cap \Pi'$, and a fan is a compactifying fan for $\mathcal{P} \cap \mathcal{P}'$ if and only it is a compactifying fan for $\Pi \cap \Pi'$.

Remark 3.3. Let $\mathcal{P}$ be a finite collection of polyhedra in $N_{\mathbf{R}}$ and let Δ be a compactifying fan for $\mathcal{P}$. We claim that the closure of $|\mathcal{P}|$ in $N_{\mathbf{R}}(\Delta)$ is compact. To prove this we may assume that $\mathcal{P} = \{P\}$ consists of a single polyhedron, and by Lemma 4.4 below we may even assume that $\rho(P) \in \Delta$. The closure of P in $N_{\mathbf{R}}(\rho(P))$ is compact by [Rab10, §3], so the claim follows since $N_{\mathbf{R}}(\rho(P))$ is a subspace of $N_{\mathbf{R}}(\Delta)$.

Definition 3.4. Let $a, b \in \mathbf{R}$ with $a \leq b$ and let V be a finite-dimensional real vector space. A *continuous family of polyhedra* in V, parameterized by $[a, b]$, is a function $\mathcal{P}$ from $[a, b]$ to the set of all polyhedra in V, given by an equation of the form

$$\mathcal{P}(t) = \bigcap_{i=1}^{m} \{ v \in N_{\mathbf{R}} : \langle u_i, v \rangle \leq f_i(t) \},$$

where $u_i \in V^*$ for $i = 1, \ldots, m$, and $f_i(t)$ a continuous real-valued function on $[a, b]$.

Note that in the above definition, we allow $V = (0)$, in which case each u_i is necessarily 0, and each $\mathcal{P}(t)$ is either empty or V according to whether or not all the $f_i(t)$ are nonnegative. In addition, we allow $a = b$, in which case $\mathcal{P}$ is just a polyhedron. Note also that if $\mathcal{P}, \mathcal{P}'$ are continuous families of polyhedra in V parameterized by $[a, b]$ then $t \mapsto \mathcal{P}(t) \cap \mathcal{P}'(t)$ is one also.

For the convenience of the reader we include proofs of the following lemmas on polyhedra, which are undoubtedly well known. The first lemma roughly says that if P is a polyhedron, then we have $\lim_{t \to 0}(tP) = \rho(P)$.

Lemma 3.5. *Let V be a finite-dimensional real vector space and let $P \subseteq V$ be a polyhedron. For $t \in [0, 1]$ define*

$$\mathcal{P}(t) = \begin{cases} tP & t \in (0, 1] \\ \rho(P) & t = 0. \end{cases}$$

Then $\mathcal{P}$ is a continuous family of polyhedra.

Proof. Suppose that P is defined by the inequalities $\langle u_i, v \rangle \leq c_i$ for some $u_1, \ldots, u_m \in M_{\mathbf{R}}$ and $c_1, \ldots, c_m \in \mathbf{R}$. Then tP is defined by $\langle u_i, v \rangle \leq t c_i$ for $i = 1, \ldots, m$, so the lemma follows because $\rho(P)$ is given by $\langle u_i, v \rangle \leq 0$ for $i = 1, \ldots, m$. ■

Lemma 3.6. *Given a finite-dimensional real vector space V and a continuous family of polyhedra $\mathcal{P}$ in V, the image of $\mathcal{P}$ under projection to any quotient space W of V is a continuous family of polyhedra.*

Here the projection is taken one t at a time, in the obvious way.

Proof. Since every projection can be factored as a composition of projections with 1-dimensional kernels, it is enough to consider this case. Accordingly, let W be a quotient of V, with the kernel of $V \twoheadrightarrow W$ being 1-dimensional. Choose a basis $x_1, \ldots, x_n$ of V^*, and write

$$u_i = \sum_{j=1}^{n} a_{i,j} x_j$$

for each i. We may further suppose that we have chosen the x_i so that the kernel of the given projection is precisely the intersection of the kernels of $x_2, \ldots, x_n$. Thus, $x_2, \ldots, x_n$ gives a basis for W^*. Without loss of generality, we may reorder the u_i so that $a_{1,1}, \ldots, a_{p,1} = 0$, $a_{p+1,1}, \ldots, a_{q,1} > 0$, and $a_{q+1,1}, \ldots, a_{m,1} < 0$. Dividing through the u_i and f_i for $i > p$ by $a_{i,1}$, we have that the inequalities defining $\mathcal{P}$ can be rewritten as follows: for $i = 1, \ldots, p$, we have $\langle u_i', v \rangle \leq f_i(t)$, where $u_i' = u_i$; for $i = p + 1, \ldots, q$, we have $\langle x_1, v \rangle \leq \langle u_i', v \rangle + f_i(t)$, where $u_i' = x_1 - u_i$; and for $i = q + 1, \ldots, m$, we have $\langle x_1, v \rangle \geq \langle u_i', v \rangle + f_i(t)$, where $u_i' = x_1 - u_i$. Noting that each u_i' is now well defined on W, we see that the image of $\mathcal{P}$ in W is described by the inequalities $\langle u_i', v \rangle \leq f_i(t)$ for $i = 1, \ldots, p$, and $\langle u_i' - u_j', v \rangle \leq f_j(t) - f_i(t)$ for each $i = q + 1, \ldots, m$ and $j = p + 1, \ldots, q$. We thus conclude the desired statement. ■

The following two corollaries of the lemma will be useful to us. Setting $a = b$ in Lemma 3.6 we have:

Corollary 3.7. *Let P be a polyhedron in $N_{\mathbf{R}}$. Then $\pi_\sigma(P)$ is a polyhedron, and in particular is closed.*

On the other hand, considering projection to the 0-space we immediately conclude:

Corollary 3.8. *The set of t for which a continuous family of polyhedra is nonempty is closed in $[a, b]$.*

Our main lemma is then the following.

Lemma 3.9. *Let P be a polyhedron in $N_{\mathbf{R}}$ and let Δ be a pointed fan. If $\overline{P}$ is the closure of P in $N_{\mathbf{R}}(\Delta)$, then*

$$\overline{P} = \coprod_{\substack{\sigma \in \Delta \\ \mathrm{relint}(\sigma) \cap \rho(P) \neq \emptyset}} \pi_{\sigma}(P).$$

Proof. Since $N_{\mathbf{R}}(\sigma)$ embeds as an open subset of $N_{\mathbf{R}}(\Delta)$ for each σ in Δ, and $N_{\mathbf{R}}(\tau)$ embeds as an open subset of $N_{\mathbf{R}}(\sigma)$ for each face τ of σ, to analyze $\overline{P}$ it suffices to consider the stratum of $\overline{P}$ lying in $N_{\mathbf{R}}/\mathrm{span}(\sigma)$ for each $\sigma \in \Delta$.

Let $\rho = \rho(P)$. First suppose $\mathrm{relint}(\sigma) \cap \rho \neq \emptyset$, and let $\overline{v} \in \pi_{\sigma}(P)$. Choose $v \in P$ such that $\pi_{\sigma}(v) = \overline{v}$, and let $w \in \mathrm{relint}(\sigma) \cap \rho$. Then $aw \in \rho$ for all $a \in \mathbf{R}_{>0}$, so $v + aw \in P$ by the definition of ρ. But $v + aw \to \overline{v}$ as $a \to \infty$ by Lemma 2.5, so we have $\overline{v} \in \overline{P}$. Hence we obtain one containment.

Next, suppose that $\overline{v} \in N_{\mathbf{R}}(\sigma)/\mathrm{span}(\sigma)$ is in $\overline{P}$. Then according to Lemma 2.5 there exists a sequence $v_1, v_2, \dots \in P$ such that $\lim_{i \to \infty} \pi_{\sigma}(v_i) = \overline{v}$, and for all $u \in \sigma^{\vee} \setminus \sigma^{\perp}$ we have $\lim_{i \to \infty} \langle u, v_i \rangle = -\infty$. In particular, we see that $\overline{v}$ is in the closure of $\pi_{\sigma}(P)$, hence in $\pi_{\sigma}(P)$ by Corollary 3.7. It is therefore enough to show that $\mathrm{relint}(\sigma) \cap \rho \neq \emptyset$.

Choose generators $u_1, \dots, u_m$ for $\sigma^{\vee}$. For $\delta \geq 0$, denote by σ_{δ} the polyhedron cut out by the conditions $\langle u_j, v \rangle \leq -1$ for $u_j \notin \sigma^{\perp}$, and $\langle u_j, v \rangle \leq \delta$ for $u_j \in \sigma^{\perp}$. Then $\sigma_0 \subseteq \mathrm{relint}(\sigma)$. Fix $\delta > 0$ and choose $\varepsilon > 0$ such that $\varepsilon \langle u_j, \overline{v} \rangle < \delta$ for all j such that $u_j \in \sigma^{\perp}$. For all $i \gg 0$ we have $\langle u_j, \varepsilon v_i \rangle \leq -1$ when $u_j \notin \sigma^{\perp}$ (since $\langle u_j, \varepsilon v_i \rangle \to -\infty$) and $\langle u_j, \varepsilon v_i \rangle \leq \delta$ when $u_j \in \sigma^{\perp}$ (since $\langle u_j, \varepsilon v_i \rangle \to \varepsilon \langle u_j, \overline{v} \rangle$). We thus see that for fixed δ and ε sufficiently small, we have $(\varepsilon P) \cap \sigma_{\delta} \neq \emptyset$. Still holding δ fixed, by Lemma 3.5 we see that

$$\varepsilon \mapsto \begin{cases} (\varepsilon P) \cap \sigma_{\delta} & \varepsilon \in (0, 1] \\ \rho \cap \sigma_{\delta} & \varepsilon = 0 \end{cases}$$

forms a continuous family of polyhedra, so by Corollary 3.8 we conclude that $\rho \cap \sigma_{\delta} \neq \emptyset$. But now letting δ vary, we have that $\rho \cap \sigma_{\delta}$ also forms a continuous family of polyhedra, so $\rho \cap \sigma_0 \neq \emptyset$, and ρ meets the relative interior of σ, as desired. ∎

Lemma 3.10. *Let $\mathcal{P}, \mathcal{P}'$ be finite collections of polyhedra and let Δ be a pointed fan in $N_{\mathbf{R}}$. If Δ is compatible with either $\mathcal{P}$ or $\mathcal{P}'$ then*

$$\overline{|\mathcal{P}| \cap |\mathcal{P}'|} = \overline{|\mathcal{P}|} \cap \overline{|\mathcal{P}'|},$$

where all closures are taken in $N_{\mathbf{R}}(\Delta)$.

Proof. We assume without loss of generality that Δ is compatible with $\mathcal{P}$. Let $P \in \mathcal{P}$ and $P' \in \mathcal{P}'$. It suffices to show that $\overline{P \cap P'} = \overline{P} \cap \overline{P'}$. First we claim that $\pi_{\sigma}(P) \cap \pi_{\sigma}(P') = \pi_{\sigma}(P \cap P')$ for all $\sigma \in \Delta$ such that $\mathrm{relint}(\sigma) \cap \rho(P) \cap \rho(P') \neq \emptyset$; note that this condition is equivalent to $\sigma \subseteq \rho(P)$ and $\mathrm{relint}(\sigma) \cap \rho(P') \neq \emptyset$. It is obvious that $\pi_{\sigma}(P) \cap \pi_{\sigma}(P') \supset \pi_{\sigma}(P \cap P')$, so let $\overline{v} \in \pi_{\sigma}(P) \cap \pi_{\sigma}(P')$. Choose $v \in P$ and $v' \in P'$ such that $\pi_{\sigma}(v) = \pi_{\sigma}(v') = \overline{v}$, and choose $w' \in \mathrm{relint}(\sigma) \cap \rho(P')$. For any $a \in \mathbf{R}$ we have $v' + aw' - v \in \mathrm{span}(\sigma)$, so for $a \gg 0$ we have $v' + aw' - v \in \sigma \subseteq \rho(P)$. Choose such an a, and set $w = v' + aw' - v$. Then $v + w = v' + aw'$; since $v + w \in P$ and $v' + aw' \in P'$, this shows that

$$\overline{v} = \pi_{\sigma}(v + w) = \pi_{\sigma}(v' + aw') \in \pi_{\sigma}(P \cap P').$$

By Lemma 3.9 as applied to P and P', we have

$$\overline{P} \cap \overline{P'} = \coprod_{\substack{\sigma \in \Delta \\ \sigma \subseteq \rho(P)}} \pi_{\sigma}(P) \cap \coprod_{\substack{\sigma \in \Delta \\ \mathrm{relint}(\sigma) \cap \rho(P') \neq \emptyset}} \pi_{\sigma}(P') = \coprod_{\substack{\sigma \in \Delta \\ \mathrm{relint}(\sigma) \cap \rho(P) \cap \rho(P') \neq \emptyset}} \left(\pi_{\sigma}(P) \cap \pi_{\sigma}(P') \right)$$

$$= \coprod_{\substack{\sigma \in \Delta \\ \mathrm{relint}(\sigma) \cap \rho(P \cap P') \neq \emptyset}} \left(\pi_{\sigma}(P \cap P') \right),$$

where the last equality follows from the above and the fact that for $P \cap P' \neq \emptyset$, we have $\rho(P \cap P') = \rho(P) \cap \rho(P')$. Applying Lemma 3.9 to $P \cap P'$, this last expression is precisely $\overline{P \cap P'}$. ∎

Applying Lemma 3.10 twice, we obtain the following.

Corollary 3.11. *Let $\mathcal{P}, \mathcal{P}', \mathcal{Q}$ be finite collections of polyhedra, and let Δ be a pointed fan in $N_{\mathbf{R}}$. Suppose that Δ is compatible with $\mathcal{Q}$ and with either $\mathcal{P} \cap \mathcal{Q}$ or $\mathcal{P}' \cap \mathcal{Q}$. Then*

$$\overline{|\mathcal{P}|} \cap \overline{|\mathcal{P}'|} \cap \overline{|\mathcal{Q}|} = \overline{|\mathcal{P}| \cap |\mathcal{P}'| \cap |\mathcal{Q}|},$$

where all closures are taken in $N_{\mathbf{R}}(\Delta)$.

Now we apply Corollary 3.11 to tropicalizations of subschemes. Assume that K is complete and nontrivially valued.

Proposition 3.12.

    (1) *Let X be a closed subscheme of $\mathbf{T}$ and let Δ be an integral pointed fan in $N_{\mathbf{R}}$. Let $\overline{X}$ be the closure of X in $X(\Delta)$. Then $\mathrm{Trop}(\overline{X}, \Delta)$ is the closure of $\mathrm{Trop}(X)$ in $N_{\mathbf{R}}(\Delta)$.*

    (2) *Let X, X' be closed subschemes of $\mathbf{T}$, let $\mathcal{P}$ be a finite collection of polyhedra in $N_{\mathbf{R}}$, and let Δ be a fan compatible with $\mathcal{P}$ and with either $\mathrm{Trop}(X) \cap \mathcal{P}$ or $\mathrm{Trop}(X') \cap \mathcal{P}$. Then*

$$\mathrm{Trop}(\overline{X}, \Delta) \cap \mathrm{Trop}(\overline{X}', \Delta) \cap \overline{|\mathcal{P}|} = \overline{\mathrm{Trop}(X) \cap \mathrm{Trop}(X') \cap |\mathcal{P}|}$$

    in $N_{\mathbf{R}}(\Delta)$.

Proof. The first part is [OP10, Lemma 3.1.1], and the second part follows immediately from the first part together with Corollary 3.11. ∎

Remark 3.13. If $\mathcal{P}$ is a finite collection of polyhedra, then there always exists a compactifying fan Δ for $\mathcal{P}$. Indeed, given $P_i \in \mathcal{P}$, let Δ_i be a complete fan containing $\rho(P_i)$ (see for instance [Roh11]). Let Δ be a common pointed refinement of all the Δ_i. Then according to Proposition 3.2, Δ is a compactifying fan for $\mathcal{P}$. Although Δ is complete, we may pass to a compactifying fan with minimal support by letting Δ' be the subfan of Δ consisting of all cones contained in $\rho(P)$ for some $P \in \mathcal{P}$, and it is clear that Δ' is still a compactifying fan for $\mathcal{P}$. If $\mathcal{P}$ consists of integral polyhedra, then we may choose the Δ_i and hence Δ and Δ' to be integral as well.

For the specific case of tropicalizations, we may also proceed as follows. If $X = V(f)$ is the hypersurface defined by a nonzero Laurent polynomial $f \in K[M]$, then any pointed refinement of the normal fan to the Newton polytope of f is a (complete) compactifying fan for $\mathrm{Trop}(X)$; see [Rab10, §12]. In general, one appeals to the *tropical basis theorem*, which states that there exist generators $f_1, \ldots, f_r$ of the ideal defining X such that $\mathrm{Trop}(X) = \bigcap_{i=1}^r \mathrm{Trop}(f_i)$. See [MS09, §2.5]. Any fan which simultaneously refines a compactifying fan for each $V(f_i)$ is a compactifying fan for $\mathrm{Trop}(X)$ by Proposition 3.2. As above, such a fan will be complete, but we may always pass to a suitable subfan.

4. The moving lemma

We begin this section by proving a tropical moving lemma, which roughly says that if $X, X' \subseteq \mathbf{T}$ are closed subschemes with $\mathrm{codim}(X) + \mathrm{codim}(X') = \dim(\mathbf{T})$, and if $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ is not a finite set of points, then for any connected component C of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ and generic $v \in N$, there exists a small $\varepsilon > 0$, and a neighborhood C' of C, such that for all $t \in [-\varepsilon, 0) \cup (0, \varepsilon]$, the set $(\mathrm{Trop}(X) + tv) \cap \mathrm{Trop}(X') \cap C'$ is finite, and furthermore that for all $t \in [-\varepsilon, \varepsilon]$, the intersection of the closures of $(\mathrm{Trop}(X) + tv)$, $\mathrm{Trop}(X')$, and C' is precisely the closure of $(\mathrm{Trop}(X) + tv) \cap \mathrm{Trop}(X')$.

The main point of this section is to give an analytic counterpart to this deformation, in the following sense. Let C be a connected component of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, and assume for simplicity that C is bounded. Let P be a polytope in $N_{\mathbf{R}}$ containing C in its interior and such that $\mathrm{Trop}(X) \cap \mathrm{Trop}(X') \cap P = C$. We will express the family $\{(\mathrm{Trop}(X) + tv) \cap \mathrm{Trop}(X') \cap P\}_{t \in [-\varepsilon, \varepsilon]}$ as the tropicalization of a natural family $\mathcal{Y}$ of analytic subspaces of $\mathbf{T}^{\mathrm{an}}$ parameterized by an analytic annulus $\mathcal{S}$, which we can then study with algebraic and analytic methods. The main result of this section is that $\mathcal{Y} \to \mathcal{S}$ is *proper*.

Much of the technical difficulty in this section is in treating the case when C is not bounded. This requires quite precise control over the relationships between the various polyhedra and fans which enter the picture.

4.1. The tropical moving lemma. Let P be an integral G-affine polyhedron in $N_{\mathbf{R}}$, so $P = \bigcap_{i=1}^{r}\{v \in N_{\mathbf{R}} \ : \ \langle u_i, v \rangle \leq a_i\}$ for some $u_1, \ldots, u_r \in M$ and $a_1, \ldots, a_r \in G$. As in [Rab10, §12], we define a *thickening* of P to be a polyhedron of the form

$$P' = \bigcap_{i=1}^{r}\{v \in N_{\mathbf{R}} \ : \ \langle u_i, v \rangle \leq a_i + \varepsilon\}$$

for some $\varepsilon > 0$ in G. Note that $\rho(P') = \rho(P)$ and that P is contained in the interior $(P')^{\circ}$ of P'. If $\mathcal{P}$ is a finite collection of integral G-affine polyhedra, a *thickening* of $\mathcal{P}$ is a collection of (integral G-affine) polyhedra of the form $\mathcal{P}' = \{P' \ : \ P \in \mathcal{P}\}$, where P' denotes a thickening of P.

Remark 4.2. Let P be a pointed integral G-affine polyhedron and let P' be a thickening of P. Let $\sigma = \rho(P) = \rho(P')$. Then the closure $\overline{P}$ of P in $N_{\mathbf{R}}(\sigma)$ is contained in the interior of $\overline{P'}$; see Lemma 3.9 and [Pay09, Remark 3.4]. More generally, if $\mathcal{P}$ is a finite collection of integral G-affine polyhedra with recession cones contained in a pointed fan Δ, and if $\mathcal{P}'$ is a thickening of $\mathcal{P}$, then the closure $\overline{|\mathcal{P}|}$ of $|\mathcal{P}|$ in $N_{\mathbf{R}}(\Delta)$ is contained in the interior of $\overline{|\mathcal{P}'|}$.

Definition 4.3. Let Δ be an integral pointed fan and let $\mathcal{P}$ be a finite collection of integral G-affine polyhedra in $N_{\mathbf{R}}$. A *refinement of* $\mathcal{P}$ is a finite collection of integral G-affine polyhedra $\mathcal{P}'$ such that every polyhedron of $\mathcal{P}'$ is contained in some polyhedron of $\mathcal{P}$, and every polyhedron of $\mathcal{P}$ is a union of polyhedra in $\mathcal{P}'$. A Δ-*decomposition of* $\mathcal{P}$ is a refinement $\mathcal{P}'$ of $\mathcal{P}$ such that $\rho(P) \in \Delta$ for all $P \in \mathcal{P}'$. A Δ-*thickening of* $\mathcal{P}$ is a thickening of a Δ-decomposition of $\mathcal{P}$.

If $\mathcal{P}'$ is a refinement of $\mathcal{P}$ then $|\mathcal{P}'| = |\mathcal{P}|$. If $\mathcal{P}'$ is a Δ-thickening of $\mathcal{P}$ then $\overline{|\mathcal{P}|} \subseteq \overline{|\mathcal{P}'|}^{\circ}$ by Remark 4.2.

Lemma 4.4. *Let $\mathcal{P}$ be a finite collection of integral G-affine polyhedra and let Δ be an integral compactifying fan for $\mathcal{P}$. Then there exists a Δ-decomposition $\mathcal{P}'$ of $\mathcal{P}$. If further $\mathcal{P}''$ is a finite collection of polyhedra such that Δ is compatible with $\mathcal{P} \cap \mathcal{P}''$, and $\mathcal{P}'$ is any Δ-decomposition of $\mathcal{P}$, then Δ is compatible with $\mathcal{P}' \cap \mathcal{P}''$.*

Proof. It suffices to prove the first part of the lemma when $\mathcal{P} = \{P\}$ is a polyhedron such that $\rho(P)$ is a union of cones in Δ. First suppose that P is pointed, and let P_1 be the convex hull of the vertices of P. By [Rab10, §3] we have $P = P_1 + \rho(P)$, so $P = \bigcup_{\sigma \in \Delta, \sigma \subseteq \rho(P)}(P_1 + \sigma)$. Hence it is enough to note that if F is an integral G-affine polytope and σ is an integral cone then $F + \sigma$ is an integral G-affine polyhedron with recession cone σ.

Now suppose that P is not pointed. Let $W' \subseteq \rho(P)$ be the largest linear space contained in $\rho(P)$ and let W be a complementary integral subspace in $N_{\mathbf{R}}$. Then $P \cap W$ is a pointed polyhedron with recession cone $\rho(P) \cap W$, so if P_1 is the convex hull of the vertices of $P \cap W$ then $P \cap W = P_1 + \rho(P) \cap W$. Hence $P = P_1 + \rho(P)$, so the proof proceeds as above.

For the second half of the lemma, given $P' \in \mathcal{P}'$, $P'' \in \mathcal{P}''$, and $\sigma \in \Delta$, suppose that $\mathrm{relint}(\sigma) \cap \rho(P' \cap P'') \neq \emptyset$. Then in particular $P' \cap P'' \neq \emptyset$, and $\rho(P' \cap P'') = \rho(P') \cap \rho(P'')$, so it suffices to show that $\sigma \subseteq \rho(P') \cap \rho(P'')$. Since $\mathrm{relint}(\sigma) \cap \rho(P') \cap \rho(P'') \neq \emptyset$ and $\rho(P') \in \Delta$, we have that $\sigma \subseteq \rho(P')$, so it suffices to show $\sigma \subseteq \rho(P'')$. Let $P \in \mathcal{P}$ be a polyhedron containing P'. Then $\mathrm{relint}(\sigma) \cap \rho(P \cap P'') \neq \emptyset$, so by compatibility $\sigma \subseteq \rho(P \cap P'') \subseteq \rho(P'')$, as desired. $\blacksquare$

Definition 4.5. Let X and X' be closed subschemes of $\mathbf{T}$, and fix a choice of polyhedral complex structures on $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$. Let C be a connected component of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$. A *compactifying datum* for X, X' and C consists of a pair $(\Delta, \mathcal{P})$, where $\mathcal{P}$ is a finite collection of integral G-affine polyhedra in $N_{\mathbf{R}}$ such that

$$\mathrm{Trop}(X) \cap \mathrm{Trop}(X') \cap |\mathcal{P}| = |C|,$$

and Δ is an integral compactifying fan for $\mathcal{P}$ which is compatible with $\mathrm{Trop}(X') \cap \mathcal{P}$.

The convention that Δ should be compatible specifically with $\mathrm{Trop}(X') \cap \mathcal{P}$ rather than either $\mathrm{Trop}(X) \cap \mathcal{P}$ or $\mathrm{Trop}(X') \cap \mathcal{P}$ is made out of convenience, to simplify the statements of Lemma 4.7 and Corollary 4.8 below.

Remark 4.6. If $\mathcal{P} = C$ (with the induced polyhedral complex structure), then in order for $(\Delta, \mathcal{P})$ to be a compactifying datum for X, X' and C, it suffices that Δ be an integral compactifying fan for C, since such Δ is automatically compatible with $\mathrm{Trop}(X') \cap \mathcal{P}$. In particular, by Remark 3.13 compactifying data always exist. (The extra flexibility in the choice of $\mathcal{P}$ will be used in the proof of Theorem 6.10.)

Lemma 4.7. (Tropical moving lemma) *Let X and X' be closed subschemes of $\mathbf{T}$, and suppose that* $\mathrm{codim}(X) + \mathrm{codim}(X') = \dim(\mathbf{T})$. *Choose polyhedral complex structures on $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$. Let C be a connected component of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ and let $(\Delta, \mathcal{P})$ be a compactifying datum for X, X' and C. There exists a Δ-thickening $\mathcal{P}'$ of $\mathcal{P}$, a number $\varepsilon > 0$, and a cocharacter $v \in N$ with the following properties:*

1. *$(\Delta, \mathcal{P}')$ is a compactifying datum for X, X' and C.*
2. *For all $r \in [-\varepsilon, 0) \cup (0, \varepsilon]$, the set $(\mathrm{Trop}(X) + r \cdot v) \cap \mathrm{Trop}(X') \cap |\mathcal{P}'|$ is finite and contained in $|\mathcal{P}'|^\circ$, and each point lies in the interior of facets of $\mathrm{Trop}(X) + r \cdot v$ and $\mathrm{Trop}(X')$.*

Proof. We begin with the observation that if P, P' are disjoint polyhedra then there exists a thickening of P which is disjoint from P'. Indeed, write $P = \bigcap_{i=1}^r \{v \in N_{\mathbf{R}} \ : \ \langle u_i, v \rangle \leq a_i\}$ for $u_1, \ldots, u_r \in M$ and $a_1, \ldots, a_r \in G$, and for $t \geq 0$ set $P_t = \bigcap_{i=1}^r \{v \in N_{\mathbf{R}} \ : \ \langle u_i, v \rangle \leq a_i + t\}$. Then $t \mapsto P_t \cap P'$ is a continuous family of polyhedra with $P_0 \cap P' = \emptyset$, so by Corollary 3.8 we have $P_t \cap P' = \emptyset$ for some $t > 0$.

By Lemma 4.4 there exists a Δ-decomposition $\mathcal{P}''$ of $\mathcal{P}$, and Δ is still compatible with $\mathcal{P}'' \cap \mathrm{Trop}(X')$. Now, Δ is a compactifying fan for any thickening $\mathcal{P}'$ of $\mathcal{P}''$. It follows from the above observation that $\mathcal{P}'$ may be chosen such that $\mathrm{Trop}(X) \cap \mathrm{Trop}(X') \cap |\mathcal{P}'| = C$, and such that for each polyhedron $P' \in \mathcal{P}'$, if P' is a thickening of $P'' \in \mathcal{P}''$, then P' meets precisely the same polyhedra of $\mathrm{Trop}(X')$ as P''. Given $P \in \mathrm{Trop}(X')$ meeting P'', note that $\rho(P \cap P'') = \rho(P) \cap \rho(P'') = \rho(P) \cap \rho(P') = \rho(P \cap P')$, so the compatibility of Δ with $\mathrm{Trop}(X') \cap \mathcal{P}'$ follows from the compatibility with $\mathrm{Trop}(X') \cap \mathcal{P}''$. This proves (1).

For any $v \in N$, in order to prove that there exists $\varepsilon > 0$ with $(\mathrm{Trop}(X) + r \cdot v) \cap \mathrm{Trop}(X') \cap |\mathcal{P}'| \subseteq |\mathcal{P}'|^\circ$ for all $r \in [-\varepsilon, 0) \cup (0, \varepsilon]$, we argue similarly to the above. Indeed, note that $r \mapsto (P + r \cdot v) \cap P'$ is a continuous family of polyhedra for any polyhedra $P \subseteq \mathrm{Trop}(X), P' \subseteq \mathrm{Trop}(X')$, and that $|\mathcal{P}'| \setminus |\mathcal{P}'|^\circ$ is contained in $|\mathcal{P}'| \setminus \bigcup_{P \in \mathcal{P}'} P^\circ$, which is a finite union of polyhedra disjoint from $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$. The finiteness assertion for suitable choice of v follows from the fact that $\dim(P) + \dim(P') \leq \dim(\mathbf{T})$ for any polyhedra $P \subseteq \mathrm{Trop}(X)$ and $P' \subseteq \mathrm{Trop}(X')$, since generic translates of any two affine spaces of complementary dimension intersect in one or zero points. Similarly, any point lies in the interior of facets because the lower-dimensional faces have subcomplementary dimension, and thus generic translates do not intersect. ∎

A tuple $(\mathcal{P}', \varepsilon, v)$ satisfying Lemma 4.7 will be called a set of *tropical moving data* for $(\Delta, \mathcal{P})$.

Corollary 4.8. *In the situation of Lemma 4.7, we have*

$$\overline{\mathrm{Trop}(X)} \cap \overline{\mathrm{Trop}(X')} \cap \overline{|\mathcal{P}'|} = \overline{C} \subseteq \overline{|\mathcal{P}|} \subseteq \overline{|\mathcal{P}'|}^\circ,$$

and for all $r \in [-\varepsilon, 0) \cup (0, \varepsilon]$ we have

$$\overline{\mathrm{Trop}(X) + r \cdot v} \cap \overline{\mathrm{Trop}(X')} \cap \overline{|\mathcal{P}'|} = (\mathrm{Trop}(X) + r \cdot v) \cap \mathrm{Trop}(X') \cap |\mathcal{P}'| \subseteq |\mathcal{P}'|^\circ \subseteq \overline{|\mathcal{P}'|}^\circ,$$

all closures being taken in $N_{\mathbf{R}}(\Delta)$.

Proof. This follows immediately from the compatibility hypotheses of a compactifying datum, together with Proposition 3.12, noting that $\overline{|\mathcal{P}|} \subseteq \overline{|\mathcal{P}'|}^\circ$ by Remark 4.2, and, for the second statement, that the closure of a finite set is itself. ∎

4.9. Relative boundary and properness in analytic geometry. Our next goal is to construct a proper family of analytic spaces from compactifying and moving data as above. First we briefly review the analytic notion of properness. In this section we assume that K is complete and nontrivially valued. Recall from (2.2) that by an analytic space we mean a Hausdorff, good, strictly K-analytic space.

Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of analytic spaces. There is a canonical open subset $\mathrm{Int}(\mathcal{X}/\mathcal{Y})$ of $\mathcal{X}$ called the *relative interior* of the morphism $\mathcal{X} \to \mathcal{Y}$ (not to be confused with the relative interior of a polyhedron); its complement $\partial(\mathcal{X}/\mathcal{Y})$ in $\mathcal{X}$ is the *relative boundary*. The *absolute interior* $\mathrm{Int}(\mathcal{X})$ of an analytic space $\mathcal{X}$ is the relative interior of the structure morphism $\mathcal{X} \to \mathcal{M}(K)$, and the *absolute boundary* $\partial(\mathcal{X})$ is its complement in $\mathcal{X}$. We will use the following properties of the relative interior and relative boundary; for the definition of $\mathrm{Int}(\mathcal{X}/\mathcal{Y})$ see [Ber90, §3.1].

Proposition 4.10. (Berkovich)

(1) *If $\mathcal{X}$ is an analytic domain in an analytic space $\mathcal{Y}$ then $\mathrm{Int}(\mathcal{X}/\mathcal{Y})$ coincides with the topological interior of $\mathcal{X}$ in $\mathcal{Y}$.*

(2) *Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z}$ be a sequence of morphisms of analytic spaces. Then*
$$\mathrm{Int}(\mathcal{X}/\mathcal{Z}) = \mathrm{Int}(\mathcal{X}/\mathcal{Y}) \cap f^{-1}(\mathrm{Int}(\mathcal{Y}/\mathcal{Z})).$$

(3) *Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ be morphisms of analytic spaces, let $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$, and let $f : \mathcal{X}' \to \mathcal{X}$ be the projection. Then $f^{-1}(\mathrm{Int}(\mathcal{X}/\mathcal{Y})) \subseteq \mathrm{Int}(\mathcal{X}'/\mathcal{Y}')$.*

(4) *If X is a finite-type K-scheme then $\partial(X^{\mathrm{an}}) = \emptyset$.*

See [Ber90, Proposition 3.1.3 and Theorem 3.4.1] for the proofs. The notion of a proper morphism of analytic spaces is defined in terms of the relative interior:

Definition 4.11. Let $f : \mathcal{X} \to \mathcal{Y}$ be a separated morphism of analytic spaces.

(1) f is *boundaryless* provided that $\partial(\mathcal{X}/\mathcal{Y}) = \emptyset$.
(2) f is *compact* provided that the inverse image of a compact set is compact.
(3) f is *proper* if it is both boundaryless and compact.

Proper morphisms of analytic spaces behave much like proper morphisms of schemes. A morphism $X \to Y$ of finite-type K-schemes is proper if and only if $X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is proper. A finite morphism $f : \mathcal{X} \to \mathcal{Y}$ of analytic spaces is proper. (To say that f is finite means that for every affinoid domain $\mathcal{M}(B) \subseteq \mathcal{Y}$, its inverse image $f^{-1}(\mathcal{M}(B))$ is an affinoid domain $\mathcal{M}(A)$, and A is a finite B-module.) The converse holds in the following familiar cases:

Theorem 4.12. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of analytic spaces.*

(1) *If $\mathcal{X}$ and $\mathcal{Y}$ are affinoid spaces then f is finite.*
(2) *If f has finite fibers then f is finite.*

Proof. Part (1) follows from the Kiehl's direct image theorem [Ber90, Proposition 3.3.5], and (2) is Corollary 3.3.8 of loc. cit. ∎

4.13. The tropical criterion for properness. Fix an integral pointed fan Δ in $N_{\mathbf{R}}$. Let P be an integral G-affine polyhedron in $N_{\mathbf{R}}$ with $\rho(P) \in \Delta$, and let $\overline{P}$ be its closure in $N_{\mathbf{R}}(\Delta)$. The inverse image of $\overline{P}$ under $\mathrm{trop} : X(\Delta)^{\mathrm{an}} \to N_{\mathbf{R}}(\Delta)$ is called a *polyhedral domain* and is denoted $\mathcal{U}_P$; see [Rab10, §6]. This is an affinoid domain in $X(\Delta)^{\mathrm{an}}$. If $\mathcal{P}$ is a finite collection of integral G-affine polyhedra with recession cones contained in Δ, then $\mathcal{U}_{\mathcal{P}} := \bigcup_{P \in \mathcal{P}} \mathcal{U}_P = \mathrm{trop}^{-1}(|\overline{\mathcal{P}}|)$ is a compact analytic domain in $X(\Delta)^{\mathrm{an}}$.

Lemma 4.14. *Let Δ be an integral pointed fan in $N_{\mathbf{R}}$ and let $\mathcal{P}$ be a finite collection of integral G-affine polyhedra with recession cones contained in Δ. Let $\mathcal{S}$ be an analytic space and let $p_2 : \mathcal{S} \times \mathcal{U}_{\mathcal{P}} \to \mathcal{U}_{\mathcal{P}}$ be the projection onto the second factor. Then*
$$\mathrm{Int}(\mathcal{S} \times \mathcal{U}_{\mathcal{P}}/\mathcal{S}) \supset (\mathrm{trop} \circ p_2)^{-1}(|\overline{\mathcal{P}}|^{\circ}),$$
where the closure is taken in $N_{\mathbf{R}}(\Delta)$. In particular, $\mathrm{Int}(\mathcal{U}_{\mathcal{P}}) \supset \mathrm{trop}^{-1}(|\overline{\mathcal{P}}|^{\circ})$.

Proof. By Proposition 4.10(3) we have $\mathrm{Int}(\mathcal{S} \times \mathcal{U}_{\mathcal{P}}/\mathcal{S}) \supset p_2^{-1}(\mathrm{Int}(\mathcal{U}_{\mathcal{P}}))$, so it suffices to show that $\mathrm{Int}(\mathcal{U}_{\mathcal{P}}) \supset \mathrm{trop}^{-1}(|\overline{\mathcal{P}}|^{\circ})$. By Proposition 4.10(1), $\mathrm{Int}(\mathcal{U}_{\mathcal{P}}/X(\Delta)^{\mathrm{an}})$ is the topological interior of $\mathcal{U}_{\mathcal{P}}$ in $X(\Delta)^{\mathrm{an}}$ since $\mathcal{U}_{\mathcal{P}}$ is an analytic domain in $X(\Delta)^{\mathrm{an}}$. Since $\mathrm{trop} : X(\Delta)^{\mathrm{an}} \to N_{\mathbf{R}}$ is continuous, the set $\mathrm{trop}^{-1}(|\overline{\mathcal{P}}|^{\circ})$ is open in $X(\Delta)^{\mathrm{an}}$, so $\mathrm{trop}^{-1}(|\overline{\mathcal{P}}|^{\circ}) \subseteq \mathrm{Int}(\mathcal{U}_{\mathcal{P}}/X(\Delta)^{\mathrm{an}})$. Applying Proposition 4.10(2) to

the sequence of morphisms $\mathcal{U}_\mathcal{P} \hookrightarrow X(\Delta)^{\mathrm{an}} \to \mathscr{M}(K)$, one obtains

$$\mathrm{Int}(\mathcal{U}_\mathcal{P}) = \mathrm{Int}(\mathcal{U}_\mathcal{P}/X(\Delta)^{\mathrm{an}}) \cap \mathrm{Int}(X(\Delta)^{\mathrm{an}}).$$

But $\mathrm{Int}(X(\Delta)^{\mathrm{an}}) = X(\Delta)^{\mathrm{an}}$ by Proposition 4.10(4), so $\mathrm{Int}(\mathcal{U}_\mathcal{P}) = \mathrm{Int}(\mathcal{U}_\mathcal{P}/X(\Delta)^{\mathrm{an}}) \supset \mathrm{trop}^{-1}(|\overline{\mathcal{P}}|^\circ)$. ∎

Lemma 4.15. *Let Δ be an integral pointed fan in $N_\mathbf{R}$, let $\mathcal{P}$ be a finite collection of integral G-affine polyhedra with recession cones contained in Δ, let $\mathcal{S}$ be an analytic space, and let $\mathcal{X} \subseteq \mathcal{S} \times X(\Delta)^{\mathrm{an}}$ be a Zariski-closed subspace. Suppose that $\mathrm{trop}(\mathcal{X}_s) \subseteq \overline{|\mathcal{P}|}$ for all $s \in |\mathcal{S}|$. Then $\mathcal{X} \subseteq \mathcal{S} \times \mathcal{U}_\mathcal{P}$.*

Proof. The hypothesis in the statement of the lemma is equivalent to requiring that $\mathcal{X}_s \subseteq \{s\} \times \mathcal{U}_\mathcal{P}$ for all *rigid* points $s \in |\mathcal{S}|$. Since $|\mathcal{X}|$ maps to $|\mathcal{S}|$, the set $\coprod_{s \in |\mathcal{S}|} \mathcal{X}_s \supset |\mathcal{X}|$ is everywhere dense in $\mathcal{X}$, so since $\coprod_{s \in |\mathcal{S}|} \mathcal{X}_s$ is contained in the closed subset $\mathcal{S} \times \mathcal{U}_\mathcal{P}$, we have $\mathcal{X} \subseteq \mathcal{S} \times \mathcal{U}_\mathcal{P}$. ∎

The following proposition can be found in [Rab10, §9], in a weaker form and in the language of classical rigid spaces.

Proposition 4.16. (Tropical criterion for properness) *Let Δ be an integral pointed fan in $N_\mathbf{R}$, let $\mathcal{S}$ be an analytic space, and let $\mathcal{X}$ be a Zariski-closed subspace of $\mathcal{S} \times X(\Delta)^{\mathrm{an}}$. Suppose that there exists a finite collection $\mathcal{P}$ of integral G-affine polyhedra with recession cones contained in Δ such that $\mathrm{trop}(\mathcal{X}_s, \Delta) \subseteq \overline{|\mathcal{P}|}$ for all $s \in |\mathcal{S}|$, where the closure is taken in $N_\mathbf{R}(\Delta)$. Then $\mathcal{X} \to \mathcal{S}$ is proper. Moreover, if $\mathcal{P} = \{P\}$ is a single polyhedron then $\mathcal{X} \to \mathcal{S}$ is finite.*

Proof. By Lemma 4.15, the condition on the tropicalizations implies that $\mathcal{X} \subseteq \mathcal{S} \times \mathcal{U}_\mathcal{P}$, i.e. that $\mathrm{trop}(p_2(\mathcal{X})) \subseteq \overline{|\mathcal{P}|}$. Since properness can be checked affinoid-locally on the base, we may assume that $\mathcal{S}$ is affinoid. Then $\mathcal{S} \times \mathcal{U}_\mathcal{P}$ is compact, being a finite union of affinoids, so $\mathcal{X}$ is compact, and therefore $\mathcal{X} \to \mathcal{S}$ is a compact map of topological spaces. Replacing $\mathcal{P}$ by a thickening, we can assume that $\mathrm{trop}(p_2(\mathcal{X})) \subseteq \overline{|\mathcal{P}|}^\circ$ (cf. Remark 4.2), so $\mathcal{X} \subseteq \mathrm{Int}(\mathcal{S} \times \mathcal{U}_\mathcal{P}/\mathcal{S})$ by Lemma 4.14. Applying Proposition 4.10(2) to the sequence of morphisms $\mathcal{X} \hookrightarrow \mathcal{S} \times \mathcal{U}_\mathcal{P} \to \mathcal{S}$ we obtain

$$\mathrm{Int}(\mathcal{X}/\mathcal{S}) = \mathrm{Int}(\mathcal{X}/\mathcal{S} \times \mathcal{U}_\mathcal{P}) \cap \mathrm{Int}(\mathcal{S} \times \mathcal{U}_\mathcal{P}/\mathcal{S}) = \mathrm{Int}(\mathcal{X}/\mathcal{S} \times \mathcal{U}_\mathcal{P}).$$

Since $\mathcal{X} \to \mathcal{S} \times \mathcal{U}_\mathcal{P}$ is a closed immersion it is finite, hence proper, so $\mathcal{X} = \mathrm{Int}(\mathcal{X}/\mathcal{S} \times \mathcal{U}_\mathcal{P})$; therefore $\mathrm{Int}(\mathcal{X}/\mathcal{S}) = \mathcal{X}$, so $\mathcal{X} \to \mathcal{S}$ is boundaryless and compact, hence proper.

Now suppose that $\mathcal{P} = \{P\}$ is a single polyhedron, still assuming $\mathcal{S}$ affinoid. Then $\mathcal{S} \times \mathcal{U}_\mathcal{P} = \mathcal{S} \times \mathcal{U}_P$ is affinoid, so $\mathcal{X}$ is affinoid; hence $\mathcal{X} \to \mathcal{S}$ is finite by Theorem 4.12(1), being a proper morphism of affinoids. ∎

Since properness and finiteness can be checked after analytification, we have the following algebraic consequence.

Corollary 4.17. *Let Δ be an integral pointed fan in $N_\mathbf{R}$ and let $X \subseteq X(\Delta)$ be a closed subscheme. Suppose that there exists a finite collection $\mathcal{P}$ of integral G-affine polyhedra with recession cones contained in Δ such that $\mathrm{Trop}(X, \Delta) \subseteq \overline{|\mathcal{P}|}$, where the closure is taken in $N_\mathbf{R}(\Delta)$. Then X is proper. Moreover, if $\mathcal{P} = \{P\}$ is a single polyhedron then X is finite.*

4.18. The moving construction. Finally we show how a set of tropical moving data gives rise to a proper family over an analytic annulus. Fix $X, X' \subseteq \mathbf{T}$ with $\mathrm{codim}(X) + \mathrm{codim}(X') = \dim(\mathbf{T})$ and fix a connected component C of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$. Choose polyhedral structures on $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$, let $(\Delta, \mathcal{P})$ be a compactifying datum for X, X' and C, and choose a set $(\mathcal{P}', \varepsilon, v)$ of tropical moving data for $(\Delta, \mathcal{P})$. We may assume without loss of generality that $\varepsilon \in G$. Let

$$\mathcal{S}_\varepsilon = \mathcal{U}_{[\varepsilon,\varepsilon]} = \mathrm{val}^{-1}([-\varepsilon, \varepsilon]) \subset \mathbf{G}_m^{\mathrm{an}};$$

this is the annulus whose set of $\overline{K}$-points is $\{t \in \overline{K}^\times : \mathrm{val}(t) \in [-\varepsilon, \varepsilon]\}$. It is a polytopal domain (and in particular an affinoid domain) in $\mathbf{G}_m^{\mathrm{an}}$.

Let $\overline{X}$ and $\overline{X}'$ denote the closures of X and X' in $X(\Delta)$, respectively. Considering v as a homomorphism $v : \mathbf{G}_m \to \mathbf{T}$, we obtain an action $\mu : \mathbf{G}_m \times X(\Delta) \to X(\Delta)$ of $\mathbf{G}_m$ on $X(\Delta)$ given by $\mu(t, x) = v(t) \cdot x$. Note that $(p_1, \mu) : \mathbf{G}_m \times X(\Delta) \to \mathbf{G}_m \times X(\Delta)$ is an isomorphism, where p_1 is projection onto the first factor. Let $\mathfrak{X} := (p_1, \mu)(\mathbf{G}_m \times \overline{X})$ and $\mathfrak{X}' := \mathbf{G}_m \times \overline{X}'$. These are closed subschemes of

$\mathbf{G}_m \times X(\Delta)$, which we will think of as being flat families of closed subschemes of $X(\Delta)$ parameterized by $\mathbf{G}_m$. A point $t \in \mathbf{G}_m^{\mathrm{an}}$ can be thought of as a morphism $t : \mathscr{M}(\mathscr{H}(t)) \to (\mathbf{G}_m^{\mathrm{an}})_{\mathscr{H}(t)}$, which is given by element of $\mathscr{H}(t)^\times$, and is thus the analytification of a morphism $t : \mathrm{Spec}(\mathscr{H}(t)) \to (\mathbf{G}_m)_{\mathscr{H}(t)}$. Since analytifications commute with fiber products and extension of scalars, the fiber $\mathfrak{X}_t$ of $\mathfrak{X}^{\mathrm{an}}$ over t is naturally identified with the analytification of $v(t) \cdot \overline{X}_{\mathscr{H}(t)}$, which is the closure of $v(t) \cdot X_{\mathscr{H}(t)}$ in $X(\Delta)_{\mathscr{H}(t)}$.

Let $\mathfrak{Y} = \mathfrak{X} \cap \mathfrak{X}' \subseteq \mathbf{G}_m \times X(\Delta)$ and let

$$\mathcal{Y} = \mathfrak{Y}^{\mathrm{an}} \cap (\mathcal{S}_\varepsilon \times \mathcal{U}_{\mathcal{P}'}) = \mathfrak{Y}^{\mathrm{an}} \cap (\mathrm{val} \circ p_1)^{-1}([-\varepsilon, \varepsilon]) \cap (\mathrm{trop} \circ p_2)^{-1}(\overline{|\mathcal{P}'|}).$$

This is a Zariski-closed subspace of $\mathcal{S}_\varepsilon \times \mathcal{U}_{\mathcal{P}'}$. For $t \in \mathcal{S}_\varepsilon$ we have

$$\mathrm{Trop}\left(v(t) \cdot \overline{X}_{\mathscr{H}(t)}, \Delta\right) = \mathrm{Trop}(\overline{X}, \Delta) - \mathrm{val}(t) \cdot v.$$

Proposition 4.19. *The analytic space $\mathcal{Y}$ is a union of connected components of $\mathfrak{Y}^{\mathrm{an}} \cap (\mathcal{S}_\varepsilon \times X(\Delta)^{\mathrm{an}})$. Moreover, $\mathcal{Y}$ is proper over $\mathcal{S}_\varepsilon$ and Zariski-closed in $\mathcal{S}_\varepsilon \times X(\Delta)^{\mathrm{an}}$.*

Proof. Since $\mathcal{Y}$ is the intersection of $\mathfrak{Y}^{\mathrm{an}} \cap (\mathcal{S}_\varepsilon \times X(\Delta)^{\mathrm{an}})$ with the (compact) affinoid domain $\mathcal{S}_\varepsilon \times \mathcal{U}_{\mathcal{P}'}$, it is closed. On the other hand, it follows from Corollary 4.8 and Lemma 4.15 that $\mathrm{trop}(p_2(\mathcal{Y})) \subseteq \overline{|\mathcal{P}'|}^\circ$, where $p_2 : \mathcal{S}_\varepsilon \times X(\Delta)^{\mathrm{an}} \to X(\Delta)^{\mathrm{an}}$ is projection onto the second factor. Thus, $\mathcal{Y}$ is the intersection of $\mathfrak{Y}^{\mathrm{an}} \cap (\mathcal{S}_\varepsilon \times X(\Delta)^{\mathrm{an}})$ with the open subset $(\mathrm{trop} \circ p_2)^{-1}(\overline{|\mathcal{P}'|}^\circ)$, so $\mathcal{Y}$ is both open and closed in $\mathfrak{Y}^{\mathrm{an}} \cap (\mathcal{S}_\varepsilon \times X(\Delta)^{\mathrm{an}})$. Hence $\mathcal{Y}$ is Zariski-closed in $\mathcal{S}_\varepsilon \times X(\Delta)^{\mathrm{an}}$, so $\mathcal{Y} \to \mathcal{S}_\varepsilon$ is proper by Proposition 4.16. ∎

5. Continuity of intersection numbers

In this section we prove a "continuity of intersection numbers" theorem in the context of a relative dimension-zero intersection of flat families over an analytic base. We will apply this in section 6 to the family constructed in (4.18).

In this section we assume that K is complete and nontrivially valued.

5.1. Flat and smooth morphisms of analytic spaces. We begin with a review of flatness and smoothness in analytic geometry. In general the notion of a flat morphism of analytic spaces is quite subtle. However, since we are assuming that all of our analytic spaces are strictly K-analytic, separated, and good, the situation is much simpler: a morphism $f : \mathcal{Y} \to \mathcal{X}$ of analytic spaces is *flat* provided that, for every pair of affinoid domains $\mathcal{V} = \mathscr{M}(B) \subseteq \mathcal{Y}$ and $\mathcal{U} = \mathscr{M}(A) \subseteq \mathcal{X}$ with $f(\mathcal{V}) \subseteq \mathcal{U}$, the corresponding homomorphism $A \to B$ is flat; see [Duc11, Corollary 7.2] (Ducros calls this notion "universal flatness"). This condition can be checked on an affinoid cover.

The notion of smoothness that is relevant for our purposes is called "quasi-smoothness" by Ducros in loc. cit. and "rig-smoothness" in the language of classical rigid spaces. A morphism $f : \mathcal{Y} \to \mathcal{X}$ is said to be *quasi-smooth* if it is flat with geometrically regular fibers [Duc11, Proposition 3.14].

Both flatness and quasi-smoothness are preserved under composition and change of base, and the inclusion of an analytic domain is flat and quasi-smooth. A morphism $Y \to X$ of finite-type K-schemes is flat (resp. smooth) if and only if $Y^{\mathrm{an}} \to X^{\mathrm{an}}$ is flat (resp. quasi-smooth).

Remark 5.2. The best reference for the notions of flatness and smoothness in Berkovich's language is [Duc11]; however, Ducros works in much greater generality than is necessary for our purposes. Most of the results that we will use have been known for much longer, but can only be found in the literature in the language of classical rigid spaces.

We define local intersection numbers of schemes and analytic spaces using a modification of Serre's definition:

Definition 5.3. Let Y be a smooth scheme over a field k (resp. a quasi-smooth analytic space over a nontrivially valued complete non-Archimedean field k), let $X, X' \subseteq Y$ be closed subschemes (resp. Zariski-closed subspaces), and suppose that $x \in |X \cap X'|$ is an isolated point of $X \cap X'$. The *local*

intersection number of X and X' at x is defined to be

$$i_k(x, \, X \cdot X'; \, Y) = \sum_{i=0}^{\dim(Y)} (-1)^i \dim_k \operatorname{Tor}_i^{\mathscr{O}_{Y,x}}(\mathscr{O}_{X,x}, \, \mathscr{O}_{X',x}).$$

If $X \cap X'$ is K-finite, the *intersection number* of X and X' is

$$i_k(X \cdot X'; \, Y) = \sum_{x \in |X \cap X'|} i_k(x, \, X \cdot X'; \, Y).$$

Remark 5.4. The dimension of $X \cap X'$ is zero at an isolated point x of $X \cap X'$. Hence $\mathscr{O}_{X \cap X', x}$ is an Artin local ring, being Noetherian of Krull dimension zero. The finitely generated $\mathscr{O}_{Y,x}$-module $\operatorname{Tor}_i^{\mathscr{O}_{Y,x}}(\mathscr{O}_{X,x}, \, \mathscr{O}_{X',x})$ is naturally an $\mathscr{O}_{X \cap X', x}$-module, and is therefore finite-dimensional over k. Moreover, $\mathscr{O}_{Y,x}$ is a regular local ring as Y is smooth (resp. quasi-smooth) over k; hence we have $\operatorname{Tor}_i^{\mathscr{O}_{Y,x}}(\mathscr{O}_{X,x}, \, \mathscr{O}_{X',x}) = 0$ for $i > \dim(Y)$.

Remark 5.5. Suppose that $X \cap X'$ is finite. The coherent sheaf $\mathscr{T}or_i^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{O}_{X'})$ is supported on $X \cap X'$. Hence its space of global sections $\operatorname{Tor}_i^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{O}_{X'}) = \Gamma(Y, \mathscr{T}or_i^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{O}_{X'}))$ breaks up as

$$\operatorname{Tor}_i^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{O}_{X'}) = \bigoplus_{x \in |X \cap X'|} \operatorname{Tor}_i^{\mathscr{O}_{Y,x}}(\mathscr{O}_{X,x}, \, \mathscr{O}_{X',x}),$$

so it follows that

$$i_k(X \cdot X'; \, Y) = \sum_{i=0}^{\dim(Y)} (-1)^i \dim_k \operatorname{Tor}_i^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{O}_{X'}).$$

Hence our definition agrees with [OP10, Definition 4.4.1].

Remark 5.6. It is clear that $i_k(x, X \cdot X'; Y)$ is local on Y, in that it only depends on an affine (resp. affinoid) neighborhood of x.

We have the following compatibility of algebraic and analytic intersection numbers:

Proposition 5.7. *Let Y be a smooth scheme over K, let $X, X' \subseteq Y$ be closed subschemes, and let $x \in |X \cap X'|$ be an isolated point of $X \cap X'$. Then*

$$i_K(x, \, X \cdot X'; \, Y) = i_K(x, \, X^{\mathrm{an}} \cdot (X')^{\mathrm{an}}; \, Y^{\mathrm{an}})$$

under the identification of $|X \cap X'|$ with $|X^{\mathrm{an}} \cap (X')^{\mathrm{an}}|$.

Proof. By [Con99, Lemma A.1.2(2)] the local rings $\mathscr{O}_{Y,x}$ and $\mathscr{O}_{Y^{\mathrm{an}},x}$ have the same completion, so the proposition follows from Lemma 5.9 below. ∎

Our goal will be to prove the following invariance of intersection numbers in families over analytic spaces:

Proposition 5.8. *Let $\mathcal{S}$ be an analytic space, let $\mathcal{Z}$ be a quasi-smooth analytic space, and let $f : \mathcal{Z} \to \mathcal{S}$ be a quasi-smooth morphism. Let $\mathcal{X}, \mathcal{X}' \subseteq \mathcal{Z}$ be Zariski-closed subspaces, flat over $\mathcal{S}$, such that $\mathcal{Y} = \mathcal{X} \cap \mathcal{X}'$ is finite over $\mathcal{S}$. Then the map*

$$s \longmapsto i_{K(s)}(\mathcal{X}_s \cdot \mathcal{X}'_s; \, \mathcal{Z}_s) \quad : \quad |\mathcal{S}| \longrightarrow \mathbf{Z}$$

is constant on connected components of $\mathcal{S}$.

We will need the following lemmas.

Lemma 5.9. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $\widehat{A}$ be its $\mathfrak{m}$-adic completion. Let M, N be finitely generated A-modules such that $\operatorname{Supp}(M) \cap \operatorname{Supp}(N) = \{\mathfrak{m}\}$. Then for all $i \geq 0$, the natural map*

$$\operatorname{Tor}_i^A(M, \, N) \longrightarrow \operatorname{Tor}_i^{\widehat{A}}(M \otimes_A \widehat{A}, \, N \otimes_A \widehat{A})$$

is an isomorphism.

Proof. Let $\mathfrak{a} = \mathrm{Ann}(M) + \mathrm{Ann}(N)$. Since $\mathfrak{m}/\mathfrak{a}$ is a nilpotent ideal in $A/\mathfrak{a}$, the finitely generated $A/\mathfrak{a}$-module $\mathrm{Tor}_i^A(M, N)$ is $\mathfrak{m}$-adically discrete, so $\mathrm{Tor}_i^A(M, N) \to \mathrm{Tor}_i^A(M, N) \otimes_A \widehat{A}$ is an isomorphism. But $A \to \widehat{A}$ is flat, so

$$\mathrm{Tor}_i^{\widehat{A}}(M \otimes_A \widehat{A}, \, N \otimes_A \widehat{A}) \cong \mathrm{Tor}_i^A(M, N) \otimes_A \widehat{A}$$

naturally.   ∎

Recall from (2.2) that by an analytic space we mean a Hausdorff, good, strictly K-analytic space.

Lemma 5.10. *Let $f : \mathcal{Z} \to \mathcal{S}$ be a morphism of analytic spaces and let $\mathcal{Y} \subseteq \mathcal{Z}$ be a Zariski-closed subspace which is finite over $\mathcal{S}$. Then for any point $s \in \mathcal{S}$, there exists an affinoid neighborhood $\mathcal{U}$ of s and an affinoid domain $\mathcal{V} \subseteq f^{-1}(\mathcal{U})$ such that $\mathcal{Y} \cap f^{-1}(\mathcal{U}) \subseteq \mathcal{V}$.*

Proof. Fix $s \in \mathcal{S}$. We may replace $\mathcal{S}$ with an affinoid neighborhood of s to assume $\mathcal{S}$ affinoid. For $y \in \mathcal{Y}_s := f^{-1}(s) \cap \mathcal{Y}$ let $\mathcal{V}(y)$ be an affinoid neighborhood of y in $\mathcal{Z}$. We may choose the $\mathcal{V}(y)$ such that $\mathcal{V}(y) \cap \mathcal{V}(y') = \emptyset$ for $y \neq y'$; this is possible because $\mathcal{Y}_s$ is a finite set of points in the Hausdorff space $\mathcal{Z}$, and the affinoid neighborhoods of a point form a base of closed neighborhoods around that point. Let $\mathcal{V}(y)^\circ$ denote the interior of $\mathcal{V}(y)$ in $\mathcal{Z}$ and let $C = f(\mathcal{Y} \setminus \bigcup_{y \in \mathcal{Y}_s} \mathcal{V}(y)^\circ)$. Since $\mathcal{S}$ is affinoid and $\mathcal{Y} \to \mathcal{S}$ is finite, $\mathcal{Y}$ is affinoid, hence compact; therefore $\mathcal{Y} \setminus \bigcup_{y \in \mathcal{Y}_s} \mathcal{V}(y)^\circ$ is compact, so C is compact, hence closed in $\mathcal{S}$. By construction, a point $s' \in \mathcal{S}$ is not contained in C if and only if $\mathcal{Y}_{s'} \subseteq \bigcup_{y \in \mathcal{Y}_s} \mathcal{V}(y)^\circ$; in particular, $s \notin C$. Let $\mathcal{U}$ be an affinoid neighborhood of s contained in $\mathcal{S} \setminus C$ and let $\mathcal{V} = f^{-1}(\mathcal{U}) \cap \bigcup_{y \in \mathcal{Y}_s} \mathcal{V}(y)$. Clearly $f^{-1}(\mathcal{U}) \cap \mathcal{Y} \subseteq \mathcal{V}$. Since the $\mathcal{V}(y)$ are disjoint, the union $\bigcup_{y \in \mathcal{Y}_s} \mathcal{V}(y)$ is affinoid, so $\mathcal{V}$ is affinoid, being a fiber product of affinoids.   ∎

Proof of Proposition 5.8.   The question is local on $\mathcal{S}$, in the following sense. The analytic space $\mathcal{S}$ is connected if and only if the associated classical rigid-analytic space $|\mathcal{S}|$ is connected — in other words, if and only if the set $|\mathcal{S}|$ is connected with respect to the Grothendieck topology generated by subsets of the form $|\mathcal{U}|$ for $\mathcal{U} \subseteq \mathcal{S}$ affinoid, with coverings being the so-called admissible coverings.[3] Concretely, this means that if we can cover $\mathcal{S}$ by affinoid domains $\{\mathcal{S}_i\}_{i \in I}$ such that every point of $\mathcal{S}$ is contained in the interior of some $\mathcal{S}_i$, then it suffices to prove the proposition after base change to each $\mathcal{S}_i$. See [Ber93, §1.6]. By Lemma 5.10 we may assume that $\mathcal{S} = \mathscr{M}(R)$ and $\mathcal{Z} = \mathscr{M}(C)$ are affinoid, and that $\mathcal{S}$ is connected. Hence $\mathcal{X} = \mathscr{M}(A)$, $\mathcal{X}' = \mathscr{M}(A')$, and $\mathcal{Y} = \mathscr{M}(B)$ affinoid as well.

Now we proceed as in the proof of [OP10, Theorem 4.4.2]. Let $P_\bullet$ be a resolution of A by finite free C-modules and let $Q_\bullet = P_\bullet \otimes_C A'$. Then the homology of $Q_\bullet$ calculates the groups $\mathrm{Tor}_i^C(A, A')$. For any maximal ideal $\mathfrak{p} \subset C$ the localization of $\mathrm{Tor}_i^C(A, A')$ at $\mathfrak{p}$ is canonically isomorphic to $\mathrm{Tor}_i^{C_\mathfrak{p}}(A_\mathfrak{p}, A'_\mathfrak{p})$; since $\mathcal{Z}$ is quasi-smooth, its local rings are regular, so $C_\mathfrak{p}$ is regular [BGR84, Proposition 7.3.2/8], and hence $\mathrm{Tor}_i^{C_\mathfrak{p}}(A_\mathfrak{p}, A'_\mathfrak{p}) = 0$ for $i > \dim(\mathcal{Z})$. It follows that $\mathrm{Tor}_i^C(A, A') = 0$ for $i > \dim(\mathcal{Z})$.

Let $s \in |\mathcal{S}|$ and write $B_s = B \otimes_R K(s)$, $C_s = C \otimes_R K(s)$, etc. Since A is R-flat, $P_\bullet \otimes_R K(s)$ is a resolution of A_s by finite free C_s-modules, so $Q_\bullet \otimes_R K(s)$ computes $\mathrm{Tor}_\bullet^{C_s}(A_s, A'_s)$. Let y be a point of $\mathcal{Y}_s$, and let $\mathfrak{p}$ be the corresponding maximal ideal of C_s. By [BGR84, Proposition 7.3.2/3], the local rings $(C_s)_\mathfrak{p}$ and $\mathscr{O}_{\mathcal{Y}_s, y}$ have the same completion. Since $\mathrm{Tor}_i^{C_s}(A_s, A'_s)$ is supported on the finite set of points of $\mathcal{Y}_s$ for all i, we have

$$\mathrm{Tor}_i^{C_s}(A_s, A'_s) \cong \bigoplus_{\mathfrak{p} \in \mathcal{Y}_s} \mathrm{Tor}_i^{(C_s)_\mathfrak{p}}\big((A_s)_\mathfrak{p}, (A'_s)_\mathfrak{p}\big) \cong \bigoplus_{y \in |\mathcal{Y}_s|} \mathrm{Tor}_i^{\mathscr{O}_{\mathcal{Y}_s, y}}\big(\mathscr{O}_{\mathcal{X}_s, y}, \mathscr{O}_{\mathcal{X}'_s, y}\big),$$

where the last equality comes from Lemma 5.9. Therefore

$$\sum_{i=0}^{\infty} (-1)^i \dim_{K(s)} \mathrm{Tor}_i^{C_s}(A_s, A'_s) = i_{K(s)}(\mathcal{X}_s \cdot \mathcal{X}'_s; \mathcal{Z}_s).$$

The finite C-modules $\mathrm{Tor}_\bullet^C(A, A')$ are supported on $\mathcal{Y}$, so since $\mathcal{Y}$ is finite over $\mathcal{S}$, they are in fact finite R-modules. Viewing $Q_\bullet$ as a complex of R-modules with finitely many R-finite cohomology

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[3] It would be more elegant to prove the proposition for all points $s \in \mathcal{S}$, but in that case one would have to treat the issue of exactness of the completed tensor product.

groups, it follows from [EGAIII$_1$, Corollaire 0.11.9.2] that there exists a quasi-isomorphic bounded below complex $M_\bullet$ of free R-modules of finite (constant) rank. Furthermore $Q_\bullet$ is a complex of finite free A'-modules, hence flat R-modules, so by Remark 11.9.3 of loc. cit., for $s \in |\mathcal{S}|$ the complex $M_\bullet \otimes_R K(s)$ computes the homology of $Q_\bullet \otimes_R K(s)$, i.e. the groups $\mathrm{Tor}^{C_s}_\bullet(A_s, A'_s)$. Therefore

$$i_{K(s)}(\mathcal{X}_s \cdot \mathcal{X}'_s; \mathcal{Z}_s) = \sum_i (-1)^i \dim_{K(s)}(\mathrm{Tor}^{C_s}_i(A_s, A'_s))$$

$$= \sum_i (-1)^i \dim_{K(s)}(M_i \otimes_R K(s)) = \sum_i (-1)^i \mathrm{rank}_R(M_i)$$

is independent of $s \in |\mathcal{S}|$. ∎

6. Tropical lifting theorems

We are now in a position to prove the main theorems relating algebraic and tropical intersection multiplicities when the algebraic intersection is finite but the tropical intersection is not necessarily finite (i.e. the tropicalizations do not meet properly in the terminology of [OP10]). First we prove the theorem for intersections of two subschemes, then we extend to intersections of several subschemes. In this section, we assume that K is a possibly trivially-valued non-Archimedean field which is complete or algebraically closed.

6.1. Tropical intersection multiplicities. We begin by recalling the basic definitions of tropical intersection theory. Let $X, X' \subseteq \mathbf{T}$ be pure-dimensional closed subschemes such that $\mathrm{codim}(X) + \mathrm{codim}(X') = \dim(\mathbf{T})$. We say that $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ intersect *tropically transversely* at a point $v \in \mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ if v is isolated and lies in the interior of facets in both $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$. If $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ intersect tropically transversely at v, then the *local tropical intersection multiplicity* $i(v, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$ is defined to be $[N : N_P + N_{P'}] m(P) m(P')$ where P, P' are the facets of $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ respectively containing v, we denote by N_P (respectively, $N_{P'}$) the sublattice of N spanned by the translation of P (respectively, P') to the origin, and $m(P)$ (respectively, $m(P')$) denotes the multiplicity of P in $\mathrm{Trop}(X)$ (respectively, of P' in $\mathrm{Trop}(X')$).

Now, suppose that $\mathrm{Trop}(X)$ does not meet $\mathrm{Trop}(X')$ tropically transversely. The theory of stable tropical intersection allows us to nonetheless define nonnegative intersection multiplicities at all points of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, such that the multiplicities will be positive at only finitely many points. As in Lemma 4.7, for a fixed generic cocharacter $w \in N$, and sufficiently small $t > 0$, we will have that $(\mathrm{Trop}(X) + tw)$ intersects $\mathrm{Trop}(X')$ tropically transversely. Moreover, for t sufficiently small, which facets of $(\mathrm{Trop}(X) + tw)$ and $\mathrm{Trop}(X')$ meet one another is independent of t. For $v \in \mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, we can thus define the *local tropical intersection multiplicity* to be

$$(6.1.1) \qquad i\big(v, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')\big) = \sum_{P \ni v, P' \ni v} i\big((P + tw) \cap P', (\mathrm{Trop}(X) + tw) \cdot \mathrm{Trop}(X')\big),$$

where as above P and P' are facets of $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ respectively. The fact that this definition is independent of the choice of w is a consequence of the balancing condition for tropicalizations, or can be seen algebraically via the close relationship to the intersection theory of toric varieties; see [FS97].

For us, the relevant properties of (6.1.1) are the following, which are easy consequences of the definition:

Proposition 6.2. *Let X, X' be pure-dimensional closed subschemes of $\mathbf{T}$ of complementary codimension.*

(1) *If $\mathrm{Trop}(X)$ intersects $\mathrm{Trop}(X')$ tropically transversely at v, then the two definitions above of $i(v, \mathrm{Trop}(X), \mathrm{Trop}(X'))$ agree.*
(2) *In general, if $v \in \mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t < \delta$, every point $(P + tw) \cap P'$ occurring in (6.1.1) is within ε of v.*

With these preliminaries out of the way, our starting point is the theorem of Osserman and Payne which guarantees the compatibility of local tropical intersection multiplicities with local algebraic

intersection multiplicities, when the tropicalizations intersect properly. The following is a special case of [OP10, Theorem 5.1.1] (note that the hypothesis that X, X' are subvarieties is not used anywhere in the proof).

Theorem 6.3. *Suppose that K is algebraically closed. Let $X, X' \subseteq \mathbf{T}$ be pure-dimensional closed subschemes of complementary codimension. Let $v \in \mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ be an isolated point. Then there are only finitely many points $x \in |X \cap X'|$ with $\mathrm{trop}(x) = v$, and*

$$\sum_{\substack{x \in |X \cap X'| \\ \mathrm{trop}(x) = v}} i_K\big(x,\, X \cdot X';\, \mathbf{T}\big) = i\big(v,\, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')\big).$$

We extend the above theorem to a higher-dimensional connected component of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ as follows.

Theorem 6.4. *Let X, X' be pure-dimensional closed subschemes of $\mathbf{T}$ of complementary codimension. Choose polyhedral complex structures on $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$. Let C be a connected component of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ and let $(\Delta, \mathcal{P})$ be a compactifying datum for X, X' and C such that $X(\Delta)$ is smooth. Let $\overline{X}, \overline{X}'$ be the closures of X, X' in $X(\Delta)$, respectively, and let $\overline{C}$ be the closure of C in $N_{\mathbf{R}}(\Delta)$. If there are only finitely many points $x \in |\overline{X} \cap \overline{X}'|$ with $\mathrm{trop}(x) \in \overline{C}$ then*

$$(\textbf{6.4.1}) \qquad \sum_{\substack{x \in |\overline{X} \cap \overline{X}'| \\ \mathrm{trop}(x) \in \overline{C}}} i_K\big(x,\, \overline{X} \cdot \overline{X}';\, X(\Delta)\big) = \sum_{v \in C} i\big(v,\, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X')\big).$$

Proof. Let K' be a complete, nontrivially valued, algebraically closed valued field extension of K. We claim that it is enough to prove the theorem after extending to K'. Since the weights on $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ are insensitive to valued field extensions, the same is true for the tropical intersection multiplicities, so we need only show that the appropriate sums of the local algebraic intersection multiplicities are preserved after extending scalars to K'. Given $v \in \overline{C}$, let U be an open subscheme of $X(\Delta)$ such that $|\overline{X} \cap \overline{X}' \cap U|$ is the set of all points of $|\overline{X} \cap \overline{X}'|$ tropicalizing to v. Then
$(\textbf{6.4.2})$

$$\sum_{\substack{x \in |\overline{X} \cap \overline{X}'| \\ \mathrm{trop}(x) = v}} i_K\big(x,\, \overline{X} \cdot \overline{X}';\, X(\Delta)\big) = i_K\big((\overline{X} \cap U) \cdot (\overline{X}' \cap U);\, U\big) = \sum_{i=0}^{\dim(\mathbf{T})} (-1)^i \dim_K \mathrm{Tor}_i^{\mathscr{O}_U}\big(\mathscr{O}_{\overline{X} \cap U},\, \mathscr{O}_{\overline{X}' \cap U}\big).$$

The right side of the above equation is visibly insensitive to field extensions. Since $\overline{X}_{K'} \cap \overline{X}'_{K'} \cap U_{K'} = (\overline{X} \cap \overline{X}' \cap U)_{K'}$ is a finite subscheme and $\mathrm{trop}(x) = v$ for all $x \in |(\overline{X} \cap \overline{X}' \cap U)_{K'}|$, we can apply (6.4.2) again after extending scalars to obtain the desired compatibility. We thus replace K with K', and assume that K is both algebraically closed and complete with respect to a nontrivial valuation.

Now, let $(\mathcal{P}', \varepsilon, v)$ be a set of tropical moving data for $(\Delta, \mathcal{P})$ as in Lemma 4.7, with $\varepsilon \in G$. Let $Z = \mathbf{G}_m \times X(\Delta)$, and let $\mathfrak{X}, \mathfrak{X}' \subseteq Z$ be the closed subschemes defined in (4.18). Carrying out the construction of (4.18), we let $\mathcal{S} = \mathcal{S}_\varepsilon \subset \mathbf{G}_m^{\mathrm{an}}$, $\mathfrak{Y} = \mathfrak{X} \cap \mathfrak{X}'$, and $\mathcal{Y} = \mathfrak{Y}^{\mathrm{an}} \cap (\mathcal{S} \times \mathcal{U}_{\mathcal{P}'})$. By Proposition 4.19, $\mathcal{Y}$ is a union of connected components of $\mathfrak{Y}^{\mathrm{an}} \cap (\mathcal{S} \times X(\Delta)^{\mathrm{an}})$, $\mathcal{Y}$ is Zariski-closed in $\mathcal{S} \times X(\Delta)^{\mathrm{an}}$, and $\mathcal{Y} \to \mathcal{S}$ is proper. For $s \in \mathcal{S}$ let $\mathcal{Y}_s$ be the fiber of $\mathcal{Y}$ over s, and let $T = \{s \in \mathcal{S} \; : \; \dim(\mathcal{Y}_s) > 0\}$. By construction the fiber of $\mathcal{Y}$ over $1 \in |\mathbf{G}_m|$ is equal $\{y \in (\overline{X} \cap \overline{X}')^{\mathrm{an}} \; : \; \mathrm{trop}(y) \in \overline{C}\}$, and by hypothesis $1 \notin T$. The theorem on semicontinuity of fiber dimension of morphisms of analytic spaces [Duc07, Theorem 4.9] then gives that T is a finite set of rigid points of $\mathcal{S}$. Replacing $\mathcal{S}$ with $\mathcal{S} \setminus T$, we have that $\mathcal{Y} \to \mathcal{S}$ is finite by Theorem 4.12(2).

Applying Proposition 5.8 with $\mathcal{Z} = \mathcal{S} \times \mathcal{U}_{\mathcal{P}'}$, $\mathcal{X} = \mathfrak{X}^{\mathrm{an}} \cap \mathcal{Z}$, and $\mathcal{X}' = (\mathfrak{X}')^{\mathrm{an}} \cap \mathcal{Z}$, and using the compatibility of analytic and algebraic local intersection numbers from Proposition 5.7, we obtain that

for all $s \in |\mathcal{S}|$,

$$\sum_{\substack{y \in |\overline{X} \cap \overline{X}'| \\ \mathrm{trop}(y) \in \overline{C}}} i_K\big(y, \overline{X} \cdot \overline{X}'; X(\Delta)\big)$$

$$= \sum_{\substack{y \in |(v(s) \cdot \overline{X}_{K(s)}) \cap \overline{X}'_{K(s)}| \\ \mathrm{trop}(y) \in |\overline{\mathcal{P}'}|}} i_{K(s)}\big(y, (v(s) \cdot \overline{X}_{K(s)}) \cdot \overline{X}'_{K(s)}; X(\Delta)_{K(s)}\big).$$

But if $\mathrm{val}(s) \neq 0$ then $\mathrm{Trop}(X) - \mathrm{val}(s) \cdot v$ meets $\mathrm{Trop}(X')$ properly, so by Theorem 6.3, the right side of the above equation is equal to

$$\sum_{v \in |\mathcal{P}|} i\big(v, (\mathrm{Trop}(X) - \mathrm{val}(s) \cdot v) \cdot \mathrm{Trop}(X')\big).$$

It follows from Proposition 6.2 that for $\mathrm{val}(s)$ sufficiently close to 0, the above quantity is equal to $\sum_{v \in C} i(v, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'))$, which finishes the proof. ∎

Remark 6.5. Recall from Remark 4.6 that if $\mathcal{P}$ is the polyhedral complex underlying C and Δ is any integral compactifying fan for C, then $(\mathcal{P}, \Delta)$ is a compactifying datum for X, X', and C. We will use the more general statement of Theorem 6.4 when proving Theorem 6.10 below.

Proposition 6.6 below is used to prove Proposition 6.7, which guarantees the finiteness hypothesis in Theorems 6.4 and 6.10. See Remark 6.11.

Proposition 6.6. *Let $\mathcal{P}$ be a finite collection of integral G-affine polyhedra in $N_{\mathbf{R}}$, and suppose that Δ is an integral compactifying fan for $\mathcal{P}$. Then there exists an integral G-affine pointed polyhedron P with $|\mathcal{P}| \subseteq P$ and $\rho(P) \in \Delta$ if and only if there exists $\sigma \in \Delta$ such that for all $P' \in \mathcal{P}$, the cone $\rho(P')$ is a face of σ.*

Proof. First suppose that P exists as in the statement, and set $\sigma = \rho(P)$. We then have to show that $\rho(P') \prec \sigma$ for all $P' \in \mathcal{P}$. Clearly $\rho(P') \subseteq \rho(P) = \sigma$, and $\rho(P')$ is a union of cones in Δ, each of which must then be a face of σ. We then conclude that $\rho(P')$ is a face of σ, as desired. Conversely, suppose that we have $\sigma \in \Delta$ as in the statement. Noting that $\mathcal{P}$ consists entirely of pointed polyhedra, let V be the set of all vertices of all polyhedra $P' \in \mathcal{P}$. The convex hull $\mathrm{conv}(V)$ of V is a polytope, and any pointed polyhedron is the Minkowski sum of its recession cone and the convex hull of its vertices, so $|\mathcal{P}| \subseteq P := \mathrm{conv}(V) + \sigma$. This P is an integral G-affine polyhedron with recession cone σ. ∎

Proposition 6.7. *In the situation of Theorem 6.4, suppose in addition that the equivalent conditions of Proposition 6.6 are satisfied for Δ and $\mathcal{P}$. Then there are automatically only finitely many points $x \in |\overline{X} \cap \overline{X}'|$ with $\mathrm{trop}(x) \in \overline{C}$.*

Proof. As in the proof of Theorem 6.4, it is clearly enough to consider the case that K is complete, with nontrivial valuation. Define $(\mathcal{P}', \varepsilon, v)$, and $\mathcal{Y}$ as in the proof of Theorem 6.4. Then

$$\mathcal{Y}_1 = (\overline{X} \cap \overline{X}')^{\mathrm{an}} \cap \mathcal{U}_{\mathcal{P}'} = (\overline{X} \cap \overline{X}')^{\mathrm{an}} \cap \mathcal{U}_{\mathcal{P}} = \{y \in (\overline{X} \cap \overline{X}')^{\mathrm{an}} : \mathrm{trop}(y) \in \overline{C}\}$$

is Zariski-closed in $X(\Delta)^{\mathrm{an}}$. By Proposition 6.6(2) there exists an integral G-affine pointed polyhedron P such that $\overline{|\mathcal{P}|} \subseteq \overline{P}$ and $\rho(P) \in \Delta$. The desired statement now follows from the last part of Proposition 4.16 (with $\mathcal{S} = \mathscr{M}(K)$) since $\mathrm{trop}(\mathcal{Y}_1) \subseteq \overline{P}$. ∎

We are now in a position to state some simpler variants of Theorem 6.4. However, to avoid redundancy we give the statements only in the strictly more general setting of multiple intersections.

6.8. Multiple intersections. Suppose Y is a smooth variety over K, and $X_1, \ldots, X_m \subseteq Y$ are closed subschemes of pure codimensions $c_1, \ldots, c_m$, with $\sum_i c_i = \dim Y$. Let x be an isolated point of $X_1 \cap \cdots \cap X_m$. The *local intersection number* of the X_i at x is defined to be

$$i_K\big(x, X_1 \cdots X_m; Y\big) := i_K\big(D_{Y,m}(x), D_{Y,m}(Y) \cdot (X_1 \times \cdots \times X_m); Y^m\big),$$

where $D_{Y,m} : Y \to Y^m$ denotes the m-fold diagonal.

Now suppose $Y = \mathbf{T}$, and v is a (not necessarily isolated) point of $\mathrm{Trop}(X_1) \cap \cdots \cap \mathrm{Trop}(X_m)$. Then we similarly define the *local tropical intersection multiplicity* of the $\mathrm{Trop}(X_i)$ at v to be

$$i\big(v,\, \mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_m)\big) := i\big(D_{N_{\mathbf{R}},m}(v),\, D_{N_{\mathbf{R}},m}(N_{\mathbf{R}}) \cdot (\mathrm{Trop}(X_1) \times \cdots \times \mathrm{Trop}(X_m))\big),$$

where $D_{N_{\mathbf{R}},m} : N_{\mathbf{R}} \to (N_{\mathbf{R}})^m$ again denotes the m-fold diagonal.

Remark 6.9. One can give equivalent definitions of multiple intersection numbers inductively, if one defines multiplicities of intersections along components using length at the generic point rather than dimension over K. One then has to multiply by the degree of the residue field extension to obtain our intersection numbers. The same equivalence holds also for tropical intersections; see for instance [OP10, § 5.2]. In particular, for $m = 2$ the above definitions coincide with the definitions we have already given. This is classical on the algebraic side, while for the tropical side one may reduce to the algebraic side by passing to the stars at the point in question, and expressing the tropical intersection multiplicities as algebraic intersection multiplicities in suitable toric varieties.

We have the following generalization of Theorem 6.4:

Theorem 6.10.   *Let $X_1, \ldots, X_m$ be pure-dimensional closed subschemes of $\mathbf{T}$ with $\sum_i \mathrm{codim}(X_i) = \dim(\mathbf{T})$ and $m \geq 2$. Choose polyhedral complex structures on the $\mathrm{Trop}(X_i)$. Let C be a connected component of $\bigcap_i \mathrm{Trop}(X_i)$ and suppose that Δ is an integral compactifying fan for C such that $X(\Delta)$ is smooth. Let $\overline{X}_i$ be the closure of X_i in $X(\Delta)$ for each i, and let $\overline{C}$ be the closure of C in $N_{\mathbf{R}}(\Delta)$. If there are only finitely many points $x \in |\bigcap_i \overline{X}_i|$ with $\mathrm{trop}(x) \in \overline{C}$ then*

$$\sum_{\substack{x \in |\bigcap_i \overline{X}_i| \\ \mathrm{trop}(x) \in \overline{C}}} i_K\big(x, \overline{X}_1 \cdots \overline{X}_m;\, X(\Delta)\big) = \sum_{v \in C} i\big(v,\, \mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_m)\big).$$

Furthermore, if there exists $\sigma \in \Delta$ such that $\rho(P)$ is a face of σ for every polyhedron P of C, then there are automatically only finitely many points $x \in |\bigcap_i \overline{X}_i|$ with $\mathrm{trop}(x) \in \overline{C}$.

   Proof. In this proof we closely follow the statement of Theorem 6.4, matching our construction with its hypotheses. The schemes $D_{\mathbf{T},m}(\mathbf{T})$ and $\prod_i X_i$ are pure-dimensional closed subschemes of $\mathbf{T}^m$ of complementary codimension. We have $\mathrm{Trop}(D_{\mathbf{T},m}(\mathbf{T})) = D_{N_{\mathbf{R}},m}(N_{\mathbf{R}})$, which is a single polyhedron, and $\mathrm{Trop}(\prod_i X_i) = \prod_i \mathrm{Trop}(X_i)$, which has a polyhedral complex structure induced by the polyhedral complex structures on the $\mathrm{Trop}(X_i)$. Clearly $D_{N_{\mathbf{R}},m}(C)$ is a connected component of $D_{N_{\mathbf{R}},m}(N_{\mathbf{R}}) \cap \prod_i \mathrm{Trop}(X_i) = D_{N_{\mathbf{R}},m}(\bigcap_i \mathrm{Trop}(X_i))$. We claim that (Δ^m, C^m) is a compactifying datum for $D_{\mathbf{T},m}(\mathbf{T})$, $\prod X_i$, and $D_{N_{\mathbf{R}},m}(C)$. It is clear that

$$C^m \cap D_{N_{\mathbf{R}},m}(N_{\mathbf{R}}) \cap (\mathrm{Trop}(X_1) \times \cdots \times \mathrm{Trop}(X_m)) = D_{N_{\mathbf{R}},m}(C),$$

while the fact that recession cones commute with products immediately implies that Δ^m is a compactifying fan for C^m. Finally, since $C^m \cap \mathrm{Trop}(X_1) \times \cdots \times \mathrm{Trop}(X_m) = C^m$, we have that Δ^m is compatible with $C^m \cap \mathrm{Trop}(X_1) \times \cdots \times \mathrm{Trop}(X_m)$. Note that $X(\Delta^m) = X(\Delta)^m$ is smooth when $X(\Delta)$ is smooth.

   Since $D_{X(\Delta),m}$ is a closed immersion, the closure of $D_{\mathbf{T},m}(\mathbf{T})$ in $X(\Delta)^m$ is $D_{X(\Delta),m}(X(\Delta))$, and since scheme-theoretic closure commutes with fiber products in this situation, the closure of $\prod_i X_i$ is $\prod_i \overline{X}_i$. Likewise, since $D_{N_{\mathbf{R}}(\Delta),m}$ is a closed embedding, the closure of $D_{N_{\mathbf{R}},m}(C)$ in $N_{\mathbf{R}}(\Delta)^m$ is $D_{N_{\mathbf{R}}(\Delta),m}(\overline{C})$. Hence there are only finitely many points of $\overline{|D_{\mathbf{T},m}(\mathbf{T}) \cap \prod_i X_i|} = |D_{X(\Delta),m}(\bigcap_i \overline{X}_i)|$ tropicalizing to $\overline{D_{N_{\mathbf{R}},m}(C)} = D_{N_{\mathbf{R}}(\Delta),m}(\overline{C})$. Therefore the hypotheses of Theorem 6.4 are satisfied, and the result follows.

   Finally, if there exists $\sigma \in \Delta$ such that $\rho(P)$ is a face of σ for every cell P of C, then $\rho(\prod_i P_i) = \prod_i \rho(P_i)$ is a face of σ^m for every cell $\prod_i P_i$ of C^m, so the finiteness condition follows from Proposition 6.7.   ■

Remark 6.11.   By Remark 3.13, Proposition 3.2(3), and the theorem on toric resolution of singularities, there exists an integral compactifying fan for C such that $X(\Delta)$ is smooth. The condition that $\bigcap_i \overline{X}_i$ be finite is more subtle; it can certainly happen that $\bigcap_i \overline{X}_i$ meets the boundary $X(\Delta) \setminus \mathbf{T}$ in

a positive-dimensional subset even when $\bigcap_i X_i$ is finite, if the last assertion of Theorem 6.10 is not applicable.

Remark 6.12. We can dispense with the smoothness hypothesis in Theorem 6.10 in the case of a complete intersection. More precisely, suppose that each X_i is the hypersurface cut out by a nonzero Laurent polynomial $f_i \in K[M]$. We endow each $\mathrm{Trop}(X_i)$ with its canonical polyhedral complex structure. Let $C \subset \bigcap_{i=1}^m \mathrm{Trop}(X_i)$ be a connected component with its induced polyhedral complex structure, let Δ be an integral compactifying fan for C, and let $\overline{C}$ be the closure of C in $N_{\mathbf{R}}(\Delta)$. Then each $\overline{X}_i \cap X(\sigma)$ is again cut out by a single equation for $\sigma \in \Delta$ such that $N_{\mathbf{R}}(\sigma)$ meets $\overline{C} \setminus C$: indeed, for such σ we have $\sigma \subset \rho(P)$ for some cell P of C by Lemma 3.9, so we can apply [Rab10, §12] to a cell P' of $\mathrm{Trop}(X_i)$ containing P. It follows that $\bigcap_{i=1}^m \overline{X}_i$ is a local complete intersection at all points tropicalizing to $\overline{C}$, and by Hochster's theorem the toric variety $X(\Delta)$ is Cohen-Macaulay, so it makes sense to define $i_K(\xi, \overline{X}_1 \cdots \overline{X}_m; X(\Delta)) = \dim_K(\mathscr{O}_{\bigcap_{i=1}^m \overline{X}_i, \xi})$ for any isolated point $\xi \in \bigcap_{i=1}^m \overline{X}_i$ tropicalizing to $\overline{C}$. In this case Theorem 6.10 simply strengthens [Rab10, §12] by adding more flexibility in the choice of fan and ground field, and is proved in the same way.

We have the following important special case, in which no compactification is needed:

Corollary 6.13. *Let $X_1, \ldots, X_m$ be pure-dimensional closed subschemes of $\mathbf{T}$ with $\sum_i \mathrm{codim}(X_i) = \dim(\mathbf{T})$. Let C be a connected component of $\bigcap_i \mathrm{Trop}(X_i)$, and suppose that C is bounded. Then there are only finitely many points $x \in |\bigcap_i X_i|$ with $\mathrm{trop}(x) \in C$, and*

$$\sum_{\substack{x \in |\bigcap_i X_i| \\ \mathrm{trop}(x) \in C}} i_K(x, X_1 \cdots X_m; \mathbf{T}) = \sum_{v \in C} i(v, \mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_m)).$$

Proof. Apply Theorem 6.10 with $\Delta = \{\{0\}\}$. ■

Remark 6.14. Suppose that K is trivially-valued. If $X \subset X(\Delta)$ is a closed subscheme then $\mathrm{trop}(x) \in \coprod_{\sigma \in \Delta} \pi_\sigma(0) \subset N_{\mathbf{R}}(\Delta)$ for every $x \in |X|$. In particular, the compactification required for Theorem 6.10 is still necessary in this situation. On the other hand, Corollary 6.13 is exactly the same as Theorem 6.3 in the trivially-valued case since if C is bounded then $C = \{0\}$.

7. AN EXAMPLE

The following example is meant to illustrate Theorem 6.4. Assume that K is complete and non-trivially valued. Let $M = N = \mathbf{Z}^3$ and let $x = x^{-e_1}, y = x^{-e_2}, z = x^{-e_3}$, where e_1, e_2, e_3 is the standard basis of $\mathbf{Z}^3$. We have $\mathbf{T} = \mathrm{Spec}(K[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]) \cong \mathbf{G}_m^3$, and for $\xi \in |\mathbf{T}|$ we have $\mathrm{trop}(\xi) = (\mathrm{val}(x(\xi)), \mathrm{val}(y(\xi)), \mathrm{val}(z(\xi)))$ according to our sign conventions.

Let $X \subset \mathbf{T}$ be the curve defined by the equations

(7.0.1) $$(y-1)^2 = x(x-1)^2 \qquad (x-1)(z-1) = 0 \qquad (z-1)^2 = 0.$$

This is a slight simplification of the degeneration of a family of twisted cubic curves found in [Har77, Example III.9.8.4]. This curve has a non-reduced point at $(1,1,1)$ and is reduced everywhere else. Hence X is not a local complete intersection at $(1,1,1)$. The tropicalization of X coincides with the tropicalization of the underlying reduced curve X^{red}, which is a nodal cubic curve in the (x,y) plane; one computes $\mathrm{Trop}(X) = \mathrm{Trop}(X^{\mathrm{red}})$ using the Newton polytope of the defining equation $(y-1)^2 = x(x-1)^2$. The tropicalization equal to the union of the rays $R_1 = \mathbf{R}_{\geq 0} \cdot e_1$, $R_2 = \mathbf{R}_{\geq 0} \cdot e_2$, and $R_3 = \mathbf{R}_{\geq 0} \cdot (-2e_1 - 3e_2)$; these rays have tropical multiplicities 2, 3, and 1, respectively. See Figure 1.

Let $X'_a \subset \mathbf{T}$ be the plane defined by $y = a$ for $a \in K$ with $\mathrm{val}(a) = 0$. Then $\mathrm{Trop}(X'_a)$ is the plane spanned by e_1 and e_3, and $\mathrm{Trop}(X) \cap \mathrm{Trop}(X'_a) = R_1$. The intersection of $\mathrm{Trop}(X'_a) + \varepsilon e_2$ with $\mathrm{Trop}(X)$ is the point εe_2 counted with multiplicity 3; hence the stable tropical intersection $\mathrm{Trop}(X) \cdot \mathrm{Trop}(X'_a)$ is the point 0 counted with multiplicity 3.
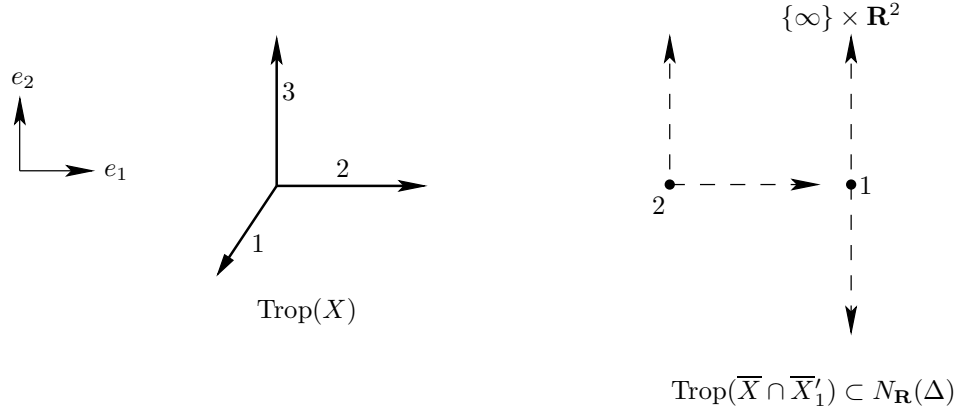
FIGURE 1. On the left: the tropicalization of the curve X defined by (7.0.1), which is contained in $\mathrm{span}(e_1, e_2)$; the e_3 direction is orthogonal to the page. On the right is the tropicalization of $\overline{X} \cap \overline{X}'_1$, with the numbers indicating the algebraic intersection multiplicities.

Let $\Delta = \{\{0\}, R_1\}$. This is a compactifying fan for R_1. We have $N_{\mathbf{R}}(\Delta) = N_{\mathbf{R}} \amalg (N_{\mathbf{R}}/\mathrm{span}(e_1))$ and $X(\Delta) \cong \mathrm{Spec}(K[x, y^{\pm 1}, z^{\pm 1}]) \cong \mathbf{A}^1 \times \mathbf{G}_m^2$; if we identify $N_{\mathbf{R}}/\mathrm{span}(e_1)$ with $\{\infty\} \times \mathbf{R}^2$ then the tropicalization map $\mathrm{trop} : |X(\Delta)| \to N_{\mathbf{R}}(\Delta)$ again can be written $\mathrm{trop}(\xi) = (\mathrm{val}(x(\xi)), \mathrm{val}(y(\xi)), \mathrm{val}(z(\xi)))$, since $\mathrm{val}(0) = \infty$. The closure $\overline{R}_1$ of R_1 in $N_{\mathbf{R}}(\Delta)$ is $R_1 \amalg \{(\infty, 0, 0)\}$, the closure $\overline{X}$ of X in $X(\Delta)$ is also given by (7.0.1), and the closure of $\overline{X}'_a$ of X'_a in $X(\Delta)$ is also given by $\{y = a\}$.

Let us calculate $\overline{X} \cdot \overline{X}'_a$. The scheme-theoretic intersection $\overline{X} \cap \overline{X}'_a$ is defined by the ideal

$$I_a = \big(x(x-1)^2 - (y-1)^2, \ (x-1)(z-1), \ (z-1)^2, \ y-a\big);$$

hence $\overline{X} \cap \overline{X}'_a$ is supported on the points of the form $(r, a, 1) \in |\mathbf{T}|$, where r is a root of the cubic polynomial $q_a(x) = x^3 - 2x^2 + x - (a-1)^2$.

Suppose first that $a = 1$, so $q_1(x) = x(x-1)^2$ and $\overline{X} \cap \overline{X}'_1$ is supported on the points $\xi_1 = (1, 1, 1)$ and $\xi_0 = (0, 1, 1)$. The point ξ_0 is reduced in $\overline{X} \cap \overline{X}'_1$ and is a smooth point of both $\overline{X}$ and $\overline{X}'_1$; hence $i_K(\xi_0, \overline{X} \cdot \overline{X}'_1; X(\Delta)) = 1$. We identify the completed local ring of $\mathbf{T}$ at ξ_1 with $B := K[\![x_1, y_1, z_1]\!]$, where $x_1 = x + 1$, $y_1 = y + 1$, and $z_1 = z + 1$. Then the completed local ring of X at ξ_1 is

$$A = K[\![x_1, y_1, z_1]\!]/(x_1^2(x_1 + 1), \ x_1 z_1, \ z_1^2) \cong K[\![x_1, y_1, z_1]\!]/(x_1^2, \ x_1 z_1, \ z_1^2),$$

and the completed local ring of X'_1 at ξ_1 is $A' = K[\![x_1, z_1]\!]$. Hence the local ring of $X \cap X'_1$ at ξ_1 is

$$A \otimes_B A' \cong K[\![x_1, y_1, z_1]\!]/(x_1^2, \ x_1 z_1, \ z_1^2, \ y_1) \cong K[\![x_1, z_1]\!]/(x_1, z_1)^2,$$

which is an Artin ring of dimension 3 over K. We have a resolution

$$0 \longrightarrow B \xrightarrow{\cdot y_1} B \longrightarrow A' \longrightarrow 0,$$

so the groups $\mathrm{Tor}_i^B(A, A')$ are calculated by the complex

$$0 \longrightarrow A \xrightarrow{\cdot y_1} A \longrightarrow 0.$$

Hence $\mathrm{Tor}_i^B(A, A') = 0$ for $i > 1$, and $\mathrm{Tor}_1^B(A, A')$ is identified with the space of y_1-torsion in A. It is not hard to see that $\mathrm{Tor}_1^B(A, A')$ is spanned over K by $y_1 z_1$, so

$$i_K(\xi_1, \overline{X} \cdot \overline{X}'_1, X(\Delta)) = \dim_K(A \otimes_B A') - \dim_K(\mathrm{Tor}_1^B(A, A')) = 3 - 1 = 2.$$

Therefore

$$3 = \sum_{v \in \overline{R}_1} i(v, \mathrm{Trop}(X) \cdot \mathrm{Trop}(X'_1)) = \sum_{\xi \in |\overline{X} \cap \overline{X}'_1|} i_K(\xi, \overline{X} \cdot \overline{X}'_1, X(\Delta)) = 2 + 1,$$

as in Theorem 6.4. Note that we would have gotten the wrong number on the right side of the above equation if we had naïvely defined the intersection number at ξ_1 as the dimension of the local ring of $X \cap X_1'$, or if we had not passed to the toric variety $X(\Delta)$ which compactifies the situation in the direction of R_1.[4]

Now suppose that $a \neq 1$ (but still $\mathrm{val}(a) = 0$). In this case $\overline{X} \cap \overline{X}_a' = X \cap X_a'$, and every point $\xi \in |X \cap X_a'|$ is a smooth point of both X_a and X_a', so $i_K(\xi, \overline{X} \cdot \overline{X}_a'; X(\Delta))$ is equal to the dimension of the local ring of $X \cap X_a'$ at ξ. Writing $\xi = (r, a, 1)$, we have $\mathrm{trop}(\xi) = (\mathrm{val}(r), 0, 0)$. The possible values for $\mathrm{val}(r)$ are easily calculated from the Newton polygon of $q_a(x) = x^3 - 2x^2 + x - (a-1)^2$; the result of this calculation is that there are two points ξ (counted with multiplicity) with $\mathrm{trop}(\xi) = (0, 0, 0)$, and one with $\mathrm{trop}(\xi) = (\mathrm{val}(a-1), 0, 0)$. In particular, $\mathrm{trop}(\xi)$ can lie *anywhere* on $\overline{R}_1 \cap N_G$, so we cannot strengthen Theorem 6.4 in such a way as to pinpoint $\mathrm{Trop}(\overline{X} \cap \overline{X}', \Delta)$ more precisely.

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[4]Coincidentally, if we had done neither of these things then the intersection numbers would coincide in this example.